Enumeration of Generalised Directed Lattice Paths in a Strip

Queen Mary
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Doctor of Philosophy

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Declaration

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Details of collaboration and publications: Chapters 3 and 4 are the basis for two separate research papers which are under preparation. The results are derived in collaboration with Thomas Prellberg.

Abstract

This thesis is about the enumeration of two models of directed lattice paths in a strip.

The first problem considered is of path diagrams formed by Dyck paths and columns underneath it, counted with respect to the length of the paths and the sum of the heights of the columns. The enumeration of these path diagrams is related to q-deformed tangent and secant numbers. Generating functions of height-restricted path diagrams are given by convergents of continued fractions. We derive expressions for these convergents in terms of basic hypergeometric functions, leading to a hierarchy of novel identities for basic hypergeometric functions. From these expressions, we also find novel expressions for the infinite continued fractions, leading to a different proof of known enumeration formulas for q-tangent and q-secant numbers.

The second problem considered is the enumeration of directed weighted paths in a strip with arbitrary step heights. Here, we find an appealing formula for their generating function in terms of a ratio of two (skew-) Schur functions, evaluated at the roots of the so-called kernel of a linear functional equation. The partitions indexing these Schur functions only depend on the size of the largest up and down steps, and the weights of the individual steps enter via the kernel roots. To aid computation, we express the skew Schur function in this formula in terms of a sum of Schur functions, and give several examples. We also consider an extension where contacts at the boundary are weighted.

Acknowledgements

First of all, I would like to begin by thanking my supervisor, Thomas Prellberg. His guidance and support has been invaluable, and the patience he has shown throughout this period has made it all possible. This work would not exist without his help.

I am grateful to Queen Mary University of London for their support throughout in form of my Research studentship and other travel funds for my visit to Austria and Scotland for conferences.

I want to thank my family for their love and support during my entire life. I would not have been able to reach this stage without them.

To my parents.

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Chapter 1

Introduction

Lattice paths can be informally described as a regular arrangement of points in a Euclidean space $\mathbb{R}^d$. Counting lattice paths is an important field in combinatorics known as Enumerative Combinatorics. This enumeration of paths is a very widely studied problem not just in combinatorics and probability theory but also in physics where the lattice paths are used to model polymers. Here we consider enumeration problems consisting of certain arrangements in $d = 2$ dimensions which are restricted by specific conditions.

There are several methods used for the enumeration of these paths. While the aim is to derive counting formulas for lattice paths with respect to fixed parameters such as total length, in some cases the enumeration of the paths can more easily be given in terms of explicit expressions for generating functions. For certain directed path problems it is possible to describe the generating functions as continued fractions [7]. One of the more modern methods for obtaining generating functions is the kernel method [2, 22].

In Chapter 2 we introduce the key terms used in the thesis, which includes Dyck paths, Motzkin paths, continued fractions, symmetric polynomials with an emphasis on Schur functions and skew Schur functions. We also introduce the methods used such as the kernel method and Cramer's rule. It also includes an overview of the generalised weighted path model and discusses surface weights to model polymer adsorption.

In Chapter 3 we investigate path diagrams bounded by Dyck paths and work out their counting using continued fraction expansions. Using the correspondence between path diagrams and continued fractions [7], we derive generating functions for restricted height path diagrams by solving the recurrence relation for the numerator and denominator of the convergents of the continued fractions. Here, the w^{th} convergent corresponds to restricting the height of the path diagrams to lie within a strip of width w. We consider two cases, differing by the maximal height of a column below a down step. These cases turn out to be intimately related to q-tangent and q-secant numbers [24], where q is the generating variable for the sum of column heights. We determine the w^{th} convergents and half plane limits of the generating functions for these path diagrams in terms of basic hypergeometric functions. This leads to certain results for q-tangent and q-secant numbers. Finally, the w^{th} convergents give interesting basic hypergeometric identities.

In Chapter 4 we establish a general relationship between the enumeration of generalised weighted paths and skew Schur functions, extending work by Bousquet-Mélou [3]. We define a model of generalised weighted paths that are directed lattice paths which can take steps out of a finite set of heights but are restricted to remain within a strip of height w and we specify start- and end-heights. We further associate weights to the height of the steps taken. Our main result comprises a theorem expressing the generating function of these paths in terms of skew Schur functions.

In Chapter 5 we extend the enumeration of these generalised weighted paths by adding contact weights at the boundaries. We thus consider the model of generalised weighted paths undergoing adsorption onto the lower and the upper boundary. Polymer adsorption has always been of interest and it has been evaluated for different lattice paths [4, 23]. We look at the special case of adsorption in Motzkin paths [5] which is then extended to generalised weighted paths.

Chapter 2

Background

This chapter discusses the key notions used in the thesis. In the next sections we introduce Dyck and Motzkin paths, and discuss their relations to the combinatorial aspects of continued fractions, as this will play a central part in Chapter 3. To prepare for Chapters 4 and 5, we introduce the kernel method as a method for solving certain linear combinatorial functional equations. As we express our results in terms of linear systems involving symmetric functions, we also briefly remind the reader of the general Laplace expansion and Cramer's rule, and summarise needed background on symmetric functions, in particular focussing on Schur functions and the Jacobi-Trudi formulas. We close this chapter by briefly discussing polymer adsorption.

2.1 Dyck paths

Definition 2.1. *A Dyck path of length n is a directed walk on $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$, which never goes below the x-axis. The step sets permitted are an up step $(1,1)$ and a down step $(1,-1)$.*

The length (no of steps) of any Dyck path is even, and Dyck paths of half length n (i.e. of length $2n$) are counted by the Catalan numbers C_n. There are various ways of obtaining a simple formula for C_n, for example by using a combinatorial construction conditioning on the first return to the x-axis: a Dyck path D is either an empty step or an up step followed by a Dyck path D_1, a down step, and another Dyck path D_2, where D_1 and D_2 may be empty. Translating this into generating functions and defining the generating function of Dyck paths to be

$$C(z) = \sum_{n=0}^{\infty} C_n z^{2n} , \tag{2.1.1}$$

we can say that $C(z)$ satisfies the quadratic equation

$$C(z) = 1 + z^2 C(z)^2 , \tag{2.1.2}$$

which can be solved to give

$$C(z) = \frac{1 \pm \sqrt{1-4z^2}}{2z^2} . \tag{2.1.3}$$

To get the postive coefficients in $C(z)$ we take the solution

$$C(z) = \frac{1 - \sqrt{1-4z^2}}{2z^2} .$$

The $2n^{th}$ coefficient of z will give us the corresponding Catalan number as follows:

$$[z^{2n}]C(z) = C_n = \frac{1}{n+1}\binom{2n}{n} . \tag{2.1.4}$$

2.2 Motzkin paths

Definition 2.2. *A Motzkin path of length n is a directed walk on $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$, which never goes below the x-axis. The step sets permitted are an up step $(1,1)$, a down step $(1,-1)$, and a horizontal step $(1,0)$.*

Thus, a Motzkin path is either an empty step or a horizontal step followed by a Motzkin path or an up step followed by a Motzkin path then a down step and then another Motzkin path. In terms of the generating function of Motzkin paths,

$$M(z) = \sum_{n=0}^{\infty} M_n z^n \ , \tag{2.2.1}$$

this implies the quadratic equation

$$M(z) = 1 + zM(z) + z^2 M(z)^2 \, . \tag{2.2.2}$$

Solving the quadratic equation gives

$$M(z) = \frac{(1-z) \pm \sqrt{(z-1)^2 - 4z^2}}{2z^2}. \tag{2.2.3}$$

For the positive coefficients we take the solution

$$M(z) = \frac{(1-z) - \sqrt{1-2z-3z^2}}{2z^2}. \tag{2.2.4}$$

The coefficient of z in the generating function $M(z)$ is the Motzkin number M_n, which thus count Motzkin paths of length n. Catalan numbers and Motzkin numbers are related by the expression

$$M_n = \sum_{k=0}^{n/2} \binom{n}{2k} C_k \ , \tag{2.2.5}$$

where the combinatorial prefactor corresponds to the number of ways horizontal steps can be inserted into a Dyck path of length $2k$ to create a Motzkin path of length n.

2.3 Continued fractions

In this section we discuss Jacobi type continued fraction expansions. We let $X = \{a_0, a_1, \ldots, b_1, b_2, \ldots, c_0, c_1, \ldots\}$ and define Jacobi type continued fraction as

$$J(X,t) = \cfrac{1}{1 - c_0 t - \cfrac{a_0 b_1 t^2}{1 - c_1 t - \cfrac{a_1 b_2 t^2}{\ddots\, a_{k-1} b_k t^2}}}. \tag{2.3.1}$$

From [7] we know that Jacobi type continued fractions have a combinatorial interpretation in terms of labelled paths in the plane as follows: a_i labels an up step starting at height i, b_i labels a down step starting at height i and c_i labels a horizontal step at height i. An example of a labelled path is shown in Figure 2.1. Finite Jacobi type

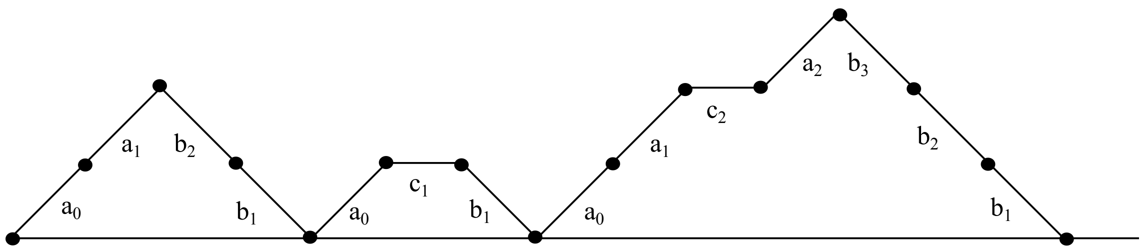

Figure 2.1: A labelled path where a_i is a label for an up step starting at height i, b_i for a down step starting at height i and c_i for a horizontal step at height i.

continued fractions as given by (2.3.1) therefore correspond to Motzkin paths in a strip of height k, whereas infinite continued fractions correspond to Motzkin paths without height restrictions. Note that if all c_i are zero then the continued fraction is simply a Stieltjes type continued fraction which represents Dyck paths.

Continued fraction expansions for Motzkin paths and Dyck paths without weights are obtained by letting $a_i = b_i = c_i = 1$; and $a_i = b_i = 1,\ c_i = 0$, respectively, and are thus given by

$$M(z) = \cfrac{1}{1 - z - \cfrac{z^2}{1 - z - \cfrac{z^2}{\ddots}}} \quad \text{and} \quad C(z) = \cfrac{1}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{\ddots}}}\,. \tag{2.3.2}$$

2.4 Path diagrams

Definition 2.3. *A system of path diagrams is defined by a possibility function*

$$pos : X \to \mathbb{N}_0.$$

Path diagrams are composed of a Motzkin path $u = u_1 u_2 u_3 \cdots u_n$, where for $j = 1, 2, \ldots, n$ each $u_j \in X$, and a corresponding sequence of integers $s = s_1 s_2 s_3 \ldots s_n$ where for $j = 1, 2, \cdots, n$ each $0 \le s_j \le pos(u_j)$. We get n points corresponding to a path of length n.

This definition has been taken from Flajolet [7], where one can also find many examples. Path diagrams have been used to enumerate various classes of permutations. They are defined by a possibility function where each particular function illustrates a different combinatorial object.

Path diagrams can be visualised by interpreting the value of the possibility function for any label as the height of a point associated to this label, as shown in Figure 2.2, where the path from Figure 2.1 has been augmented by a particular realisation of a possibility function given by

$$d = (u; s) = (a_0 a_1 b_2 b_1 a_0 c_1 b_1 a_0 a_1 c_2 a_2 b_3 b_2 b_1; 1, 1, 2, 0, 1, 0, 1, 1, 1, 2, 1, 3, 1, 0)\,. \tag{2.4.1}$$

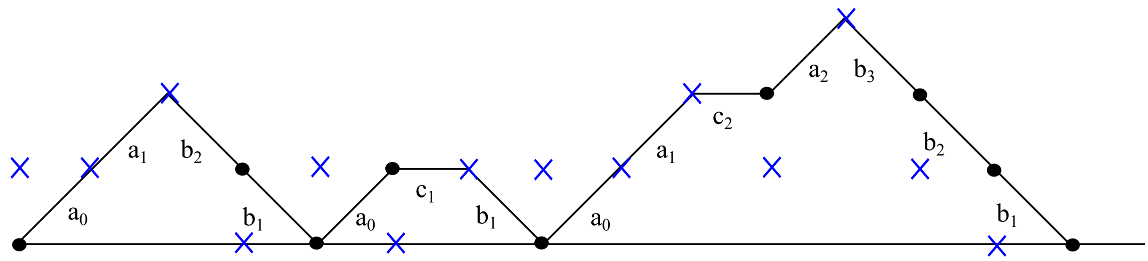

Figure 2.2: A system of path diagram given by a sequence of integers $s = (1, 1, 2, 0, 1, 0, 1, 1, 1, 2, 1, 3, 1, 0)$. Each integer represents a height of point marked by crosses.

We now move on to concepts for Chapter 4 and 5. We begin by defining the model of generalised weighted paths.

2.5 Generalised paths

Dyck paths can be generalised in many different ways. For example, increasing the step set by a horizontal step leads to Motzkin paths. Labelle and Yeh [16] considered replacing northeast steps by steps from an arbitrary finite multiset with integral coordinates and a corresponding replacement of southeast steps, which for example includes paths with chess-knight moves. Similarly, Bousquet-Mélou [3] changed the step set by expanding the vertical step size to be a subset of the integers and allowing for asymmetry between up steps and down steps. Additionally, different weights are associated to steps of different jump height [3]. Another generalisation is given by removing the restriction on starting and ending heights to be zero. For example, the paper [1] extends the work by Bousquet-Mélou by allowing the path to end at any height rather than on the x-axis.

In Chapter 4 we will introduce generalised weighted paths, which are defined in line with [3] and [1] . We will consider paths in a slit of width w starting at height u and ending at height v, which take steps from a given finite set of heights, with weights associated to these individual steps. We will call these *generalised weighted paths.*

2.6 Kernel method

In this section we talk about the kernel method, which has been used in various combinatorial problems. This orginated from Knuth's work [13] and was explained further with numerous examples given by Prodinger [22]. Some recent applications have been in [2, 10, 15].

We illustrate the kernel method by giving a simple example of non-negative lattice paths starting at the origin and taking steps $(1, 1)$ and $(1, -1)$, known as ballot paths. Let $a_n^{(i)}$ be the number of such paths from the origin to (n, i). Their generating function is given by

$$f_i(z) = \sum_{n \geq 0} a_n^{(i)} z^n \, . \tag{2.6.1}$$

These generating functions satisfy the recursions

$$f_i(z) = z f_{i-1}(z) + z f_{i+1}(z), i \geq 1 \tag{2.6.2}$$

and

$$f_0(z) = 1 + z f_1(z). \tag{2.6.3}$$

To solve this enumeration problem we introduce an additional variable x. Consider a bivariate generating function $F(z, x) = \sum_{i \geq 0} f_i(z) x^i$, which uses two generating variables that account for both the length and height of paths. Multiplying the above recursion (2.6.2) by x^i and summing, we find

$$F(z, x) = 1 + zxF(z, x) + \frac{z}{x} F(z, x) - \frac{z}{x} f_0(z) \, . \tag{2.6.4}$$

Here zx corresponds to an up step, and z/x corresponds to a down step. The kernel here is

$$K(z, x) = 1 - zx - \frac{z}{x} \, .$$

We can write $f_0(z) = F(z, 0)$, so that we get

$$F(z, x) = \frac{zF(z, 0) - x}{zx^2 - x + z} \,. \tag{2.6.5}$$

The denominator being quadratic in x can be factorised as $z(x - r_-(z))(x - r_+(z))$, where $r_\pm(z)$ are given by:

$$r_\pm(z) = \frac{1 \pm \sqrt{1 - 4z^2}}{2z} \,. \tag{2.6.6}$$

We see that $x - r_-(z) \sim x - z$ as $x, z \to 0$, so the factor $1/(x - r_-(z))$ has no power series expansion around $(0, 0)$. However $F(z, x)$ does have one, so this "bad" factor must cancel, i.e. $(x - r_-(z))$ must also be a factor of the numerator. This implies that $zF(z, 0) = r_-(z)$, and so

$$F(z, 0) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \,, \tag{2.6.7}$$

which is the well known generating function for Catalan numbers. This example illustrates the normal kernel method.

We now consider another example given in [2, eq. 11]. Here, the kernel is given by

$$K(z, u) = u^b(1 - u) + zu^b - z(1 - u) \sum_{\alpha \in A} u^{\alpha + b} + z(1 - u) \sum_{\beta \in B} u^{b - \beta} \,, \tag{2.6.8}$$

with finite sets $A \subset \mathbb{Z}$ and $B \subset \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ that specify the allowed forward jumps and the forbidden backward jumps, respectively. Letting the kernel equal zero now gives $a + b + 1$ solutions, and we are interested in the type of roots obtained. If the $a + b + 1$ solutions are expanded around 0, the roots are classified by considering their Puiseux expansion [2] as

- the unit branch, denoted by u_0, is a power series in z with constant term 1;

- b small branches, denoted by $u_1, \ldots, u_b$, are power series in $z^{1/b}$ whose first non zero term is $\varsigma z_1/b$, with $\varsigma^b + 1 = 0$;
- a large branches, denoted by $v_1, \ldots, v_a$, are Laurent series in $z^{1/a}$ whose first non zero term is $\varsigma z^{1/a}$,with $\varsigma^a + 1 = 0$.

The reason for looking at the roots is to show that all the roots are distinct. In Chapter 4 we have a kernel with $\alpha + \beta$ roots with α large roots and β small roots. Of particular interest is that roots have been translated into elementary symmetric functions and are later expressed in terms of Schur functions.

2.7 Cramer's rule

Cramer's rule is a well known method for solving a system of linear equations with a unique solution. It expresses the solution in terms of determinants. Consider a system of n linear equations with n unknowns written in matrix form as

$$Ax = b \ , \tag{2.7.1}$$

where x is the column vector with all the unknowns. Cramer's rule states that the individual values for the unknowns are given by

$$x_i = \frac{|A_i|}{|A|} \qquad i = 1, \ldots, n \ , \tag{2.7.2}$$

where A_i is the matrix formed by replacing the i^{th} column of A by the column vector b. Similar to the work in [15], we are using this method in Chapters 4 and 5 to solve the system of equations where the unknowns are generating functions of paths.

The resulting determinants can be evaluated using the Laplace expansion [20] with respect to a chosen row or column. Due to the structure of the matrices we consider, we will also make use of a generalised form of Laplace expansion where the determinant is computed by using more than one row or column. Here we follow notation used in [28]. Let $B = [b_{ij}]$ be an $n \times n$ matrix, S the set of k-element subsets of $\{1, 2, \ldots, n\}$ and H an element in it. Then the determinant of B can be expanded along the k rows identified by H as follows:

$$|B| = \sum_{L \in S} \varepsilon^{H,L} b_{H,L} c_{H,L} \ , \tag{2.7.3}$$

where $\varepsilon^{H,L}$ is the sign of the permutation determined by H and L, equal to $(-1)^{(\sum_{h \in H} h) + (\sum_{\ell \in L} \ell)}$, $b_{H,L}$ is the square minor of B obtained by deleting from B rows and columns with indices in H and L respectively, and $c_{H,L}$ (called the complement of $b_{H,L}$) is defined to be $b_{H',L'}$, with H' and L' being the complement of H and L respectively.

This generalised version of Laplace expansion has been used in Chapter 5 to compute the generating functions of paths under adsorption.

2.8 Schur functions

To define Schur functions and state the Jacobi-Trudi formulas that we will make use of in Chapter 4, we need to introduce the notion of symmetric polynomials. Material in this section follows [25], however we restrict ourselves to only considering a fixed number of variables.

Let $x = (x_1, ..., x_l)$ be a set of l indeterminates. A symmetric polynomial is a polynomial in these indeterminates that is invariant under any exchange of its arguments. Symmetric polynomials form a vector space under addition and scalar multiplication, and an algebra under addition, multiplication and scalar multiplication. We denote the space of symmetric polynomials in l variables by Λ_l.

A partition $\lambda = (\lambda_1, ..., \lambda_l)$ is a weakly decreasing sequence of l non-negative integers where λ_i are called parts. There are several bases for symmetric polynomials, perhaps the simplest one being the basis of monomial symmetric functions, given by $m_\lambda(x) = \sum_\alpha x^\alpha$ where the sum ranges over all distinct permutations α of the partition λ and where we have used the notation $x^\alpha = x_1^{\alpha_1} \dots x_l^{\alpha_l}$. Elementary symmetric polynomials are special cases defined as $e_r = m_{(1^r)}$, or more explicitly as

Definition 2.4. *Let $x = (x_1, ..., x_l)$ be a set of l indeterminates. For each integer $r \geq 0$, the r-th elementary symmetric polynomial e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and*

$$e_r(x_1, \dots, x_l) = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq l} x_{i_1} x_{i_2} \dots x_{i_r}. \tag{2.8.1}$$

where $r \geq 1$ [17].

For example, for $l = 3$, $e_0(x_1, x_2, x_3) = 1$,

$$\begin{aligned} e_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \ , \\ e_2(x_1, x_2, x_3) &= x_1 x_2 + x_1 x_3 + x_2 x_3 \ , \\ e_3(x_1, x_2, x_3) &= x_1 x_2 x_3 \ , \end{aligned}$$

and $e_r(x_1, x_2, x_3) = 0$ for $r > 3$. We can extend the elementary symmetric polynomials to a basis of Λ_l by forming $e_\lambda = e_{\lambda_1} \dots e_{\lambda_l}$ for a partition λ. Another basis, and one convenient for us, is given by Schur functions.

Definition 2.5. *Let $x = (x_1, ..., x_l)$ be a set of l indeterminates, and let λ be a partition. The Schur function s_λ is defined as*

$$s_\lambda(x_1, \dots, x_l) = \frac{a_{\lambda+\delta}(x_1, \dots, x_l)}{a_\delta(x_1, \dots, x_l)} \ , \tag{2.8.2}$$

where

$$a_\mu(x_1, \dots, x_l) = \det(x_i^{\mu_j})_{i,j=1}^l$$

and $\delta = (l-1, l-2, \dots, 0)$.

We note that a_μ is an alternating polynomial which changes sign under exchange of any two variables, and that $a_\delta(x_1, \ldots, x_l) = \det(x_i^{l-j})_{i,j=1}^l = \prod_{1 \le i < j \le l} (x_i - x_j)$ is the Vandermonde determinant.

To each partition λ we can associate a Young diagram consisting of boxes arranged as left justified rows where the i^{th} row contains λ_i boxes. The number of boxes in each row gives a partition λ of a non negative integer l, the total number of boxes of the diagram . A Young diagram with boxes filled with entries is known as a Young tableau. If the entries weakly increase along each row and strictly increase down each column then it is specifically called a semistandard Young tableau.

We now define an inner product $\langle , \rangle$ on Λ_l by requiring that $\{s_\lambda\}$ is an orthonormal basis. We can now define skew Schur functions as follows.

Definition 2.6. *Given two partitions λ and μ with $\mu \subseteq \lambda$, the skew Schur function $s_{\lambda/\mu}$ is defined via*

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle \tag{2.8.3}$$

for any partition ν.

Here $\mu \subseteq \lambda$ means that the Young diagram λ contains Young diagram μ where each $\mu_i \le \lambda_i$ for all i. A skew shape λ/μ is the set of boxes that belong to the diagram of λ but not to μ as shown in Figure 2.3. Thus, a Schur function s_λ is also a skew Schur function $s_{\lambda/\mu}$ with $\mu = ()$. A major result in the theory of symmetric

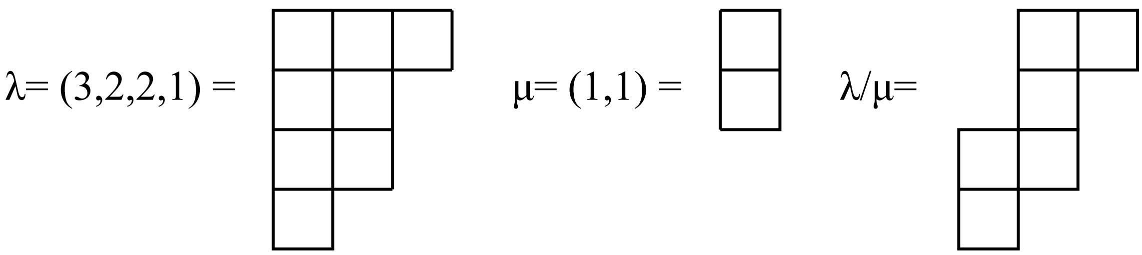

Figure 2.3: Young diagram for $s_{(3,2,2,1)/(1,1)}$

functions is that a skew Schur function $s_{\lambda/\mu}$ expands positively in Schur functions as

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu} \,, \tag{2.8.4}$$

where the coefficients $c^{\lambda}_{\mu\nu}$ are known as Littlewood-Richardson coefficients. The Littlewood-Richardson rule states that $c^{\lambda}_{\mu\nu}$ is equal to the number of semi standard Young tableaux of skew shape λ/μ of type ν whose reverse reading word is a lattice permutation, from which the positivity follows. The type of a Young tableau is defined as the tuple $(n_1, \ldots, n_k)$, where n_i is the number of occurrences of the integer i in the Young tableau. The example for Littlewood-Richardson rule is

$$s_{(3,3,2,1)/(2,1)} = s_{(2^2,1^2)} + s_{(2^3)} + s_{(3,1^3)} + 2s_{(3,2,1)} + s_{(3^2)} \,. \tag{2.8.5}$$

We have used a special case of this rule due to Pieri, where μ is a horizontal strip. Pictorially, in a horizontal strip, no two boxes of λ/μ are stacked on top of each other. This concept has been used to prove a corollary in Chapter 4.

We close this section by stating a simple identity which enables computation of Schur and skew Schur functions in terms of elementary symmetric functions. The (second) Jacobi-Trudi formula states that

$$s_{\lambda/\mu} = \det(e_{\lambda'_i - \mu'_j - i + j})_{i,j=1}^{l(\lambda)} \,, \tag{2.8.6}$$

where $l(\lambda)$ is the length of the partition λ and λ' and μ' are partitions conjugate to λ and μ, respectively. Here, a conjugate partition is most easily defined via transposition of the associated Young diagram. It is this identity via which we arrive at our main results in Chapters 4 and 5.

2.9 Polymer adsorption

We can use generalised paths to model a long chain polymer in solution. Considering paths in a strip of fixed width models polymers in a restricted domain. If there is an attractive force between the polymer and the boundary of the domain, then the polymer experiences a change in behaviour. If the attractive force is sufficiently strong, then the limiting fraction of the polymer in contact with the wall will be positive; we say that the polymer is adsorbed.

We can account for this in a path model by adding a weight every time the generalised path comes in contact with the walls. To give an example, Figure 2.4

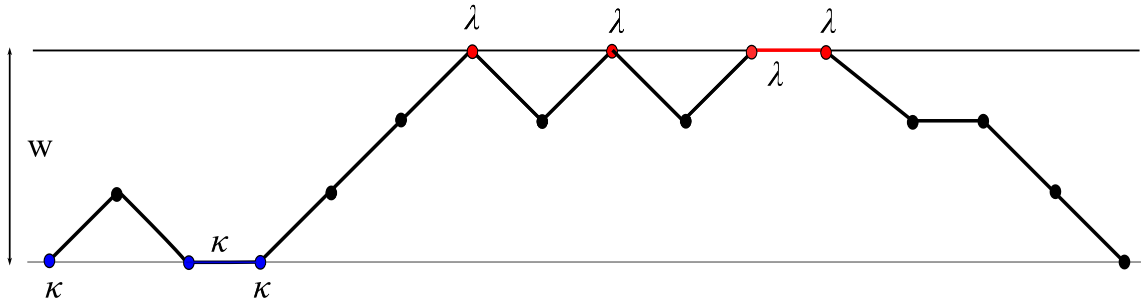

Figure 2.4: An example of a Motzkin path in slit of width w with edge and vertex contacts. The contacts at the lower and upper boundary are weighted by κ and λ respectively.

shows a Motzkin path with vertex and edge contacts at the boundaries. The model is defined to take a weight of κ whenever an up step leaves the boundary $y = 0$, and a weight of λ when a down step leaves the boundary $y = w$. When a horizontal step lies on the boundary it gets a weight of κ or λ depending on the boundary. For Motzkin paths this has been discussed in [5, 27]. In this thesis, we use this as a motivation to study an extension of generalised weighted paths where these weights have been included.

Chapter 3

Enumerating Path Diagrams

3.1 Introduction

A Dyck path is a lattice path on $\mathbb{N}^2$ from $(0, 0)$ to $(2n, 0)$ consisting of n steps in the northeast direction of the form $(1, 1)$ and n steps in the southeast direction of the form $(1, -1)$ such that the path never goes below the line $y = 0$. Figure 3.1 shows a Dyck path.

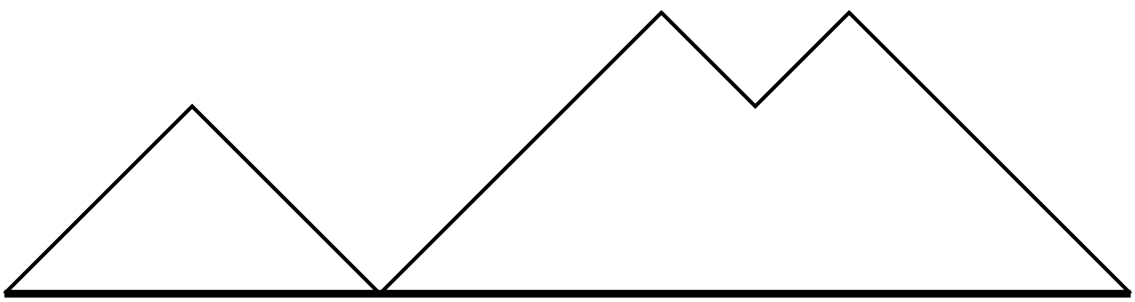

Figure 3.1: A Dyck path of half length $N = 6$.

We encode a Dyck path in terms of labelled steps where each step is indexed with the height of the point from where it starts. For example, the labelled path for Figure 3.1 would be $(a_0, a_1, b_2, b_1, a_0, a_1, a_2, b_3, a_2, b_3, b_2, b_1)$ where a_i is a northeast step starting at height i and b_j is a southeast step starting at height j. So we can say that there is a set $X = \{a_0, a_1, a_2, \ldots\} \cup \{b_1, b_2, b_3, \ldots\}$, the elements of which, as an ordered finite sequence, are associated with a Dyck path. Using the idea of defining labelled Dyck paths, we consider path diagrams which are represented by a Dyck path and the set of points under it subjected to some conditions, as introduced in Chapter 2.

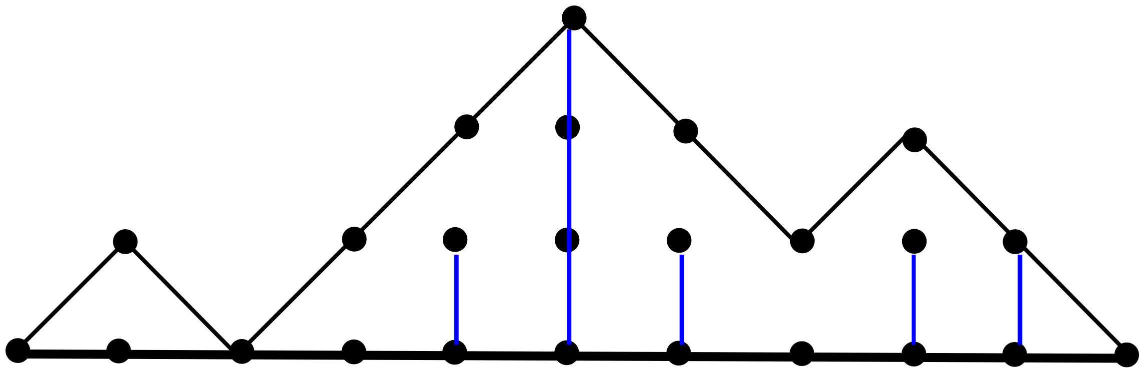

Figure 3.2: Dyck path with column heights formed by integers $s = (0, 0, 0, 0, 1, 3, 1, 0, 1, 1)$

In this chapter we consider two types of path diagrams. In the first case we consider a case in which we count all possible points bounded by a Dyck path. The possibility function is defined as

$$pos(a_j) = j, \quad pos(b_k) = k, \quad \text{for} \quad j \geq 0 \quad \text{and} \quad k \geq 1. \tag{3.1.1}$$

In the second case we restrict the set of points by eliminating the points which are in contact with the Dyck path at a southeast step. We thus have

$$pos(a_j) = j, \quad pos(b_k) = k - 1, \quad \text{for} \quad j \geq 0 \quad \text{and} \quad k \geq 1. \tag{3.1.2}$$

These possibility functions map labelled steps onto a set of integers. These integers can be visualised as column heights as in Figure 3.2.

A path is then formed by joining the peaks of the columns. For example, Figure 3.3 shows an example of one such path diagram with this path shown as

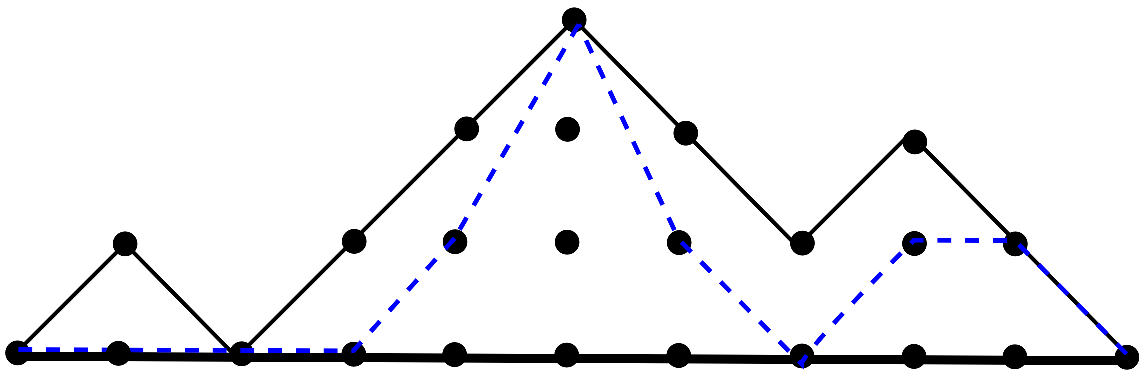

Figure 3.3: Dyck path with a path diagram

a dashed line. The corresponding set of integers belonging to this diagram are $s = (0,0,0,0,1,3,1,0,1,1)$; we can interpret this as a model of two non crossing paths in a finite slit. A lot of work has been done on the enumeration of non crossing paths in [14, 26]. The work here takes into account two paths given by a path diagram, i.e., a Dyck path and a general directed path restrained to lie between the x-axis and this Dyck path.

3.2 Generating functions and continued fraction expansions

The generating functions of path diagrams of a Dyck path weighted according to their length and the sum of column heights is given as

$$G_w(t,q) = \sum_{N,m=0}^{\infty} a_{N,m}^{(w)} t^{2N} q^m \tag{3.2.1}$$

and

$$G'_w(t,q) = \sum_{N,m=0}^{\infty} b_{N,m}^{(w)} t^{2N} q^m, \tag{3.2.2}$$

where $G_w(t,q)$ and $G'_w(t,q)$ are the generating functions weighted according to their length $2N$ and sum of column heights m. The coefficients enumerate the path diagrams for both the cases described; that is $a^{(w)}_{N,m}$ is the number of path diagrams defined by the possibility function in (3.1.1) and $b^{(w)}_{N,m}$ is the number of path diagrams formed by the possibility function in (3.1.2), bounded by a Dyck path of length $2N$ in a slit of width w. Here q is conjugate to the sum of column heights and t is conjugate to the length of the Dyck path.

Figure 3.4 shows all the possible paths of half length $N = 2$ and total column height $m = 2$. The vertical lines show the possible columns whose heights sum to 2. We find that $a^{(1)}_{2,2} = 1$ and $a^{(w)}_{2,2} = 5$ for $w \geq 2$. For $b^{(w)}_{N,m}$ the contact of path diagram and Dyck path is restricted at every southeast step, so $b^{(1)}_{2,2} = 0$ and $b^{(w)}_{2,2} = 1$ for $w \geq 2$. This corresponds to the first path in Figure 3.4.

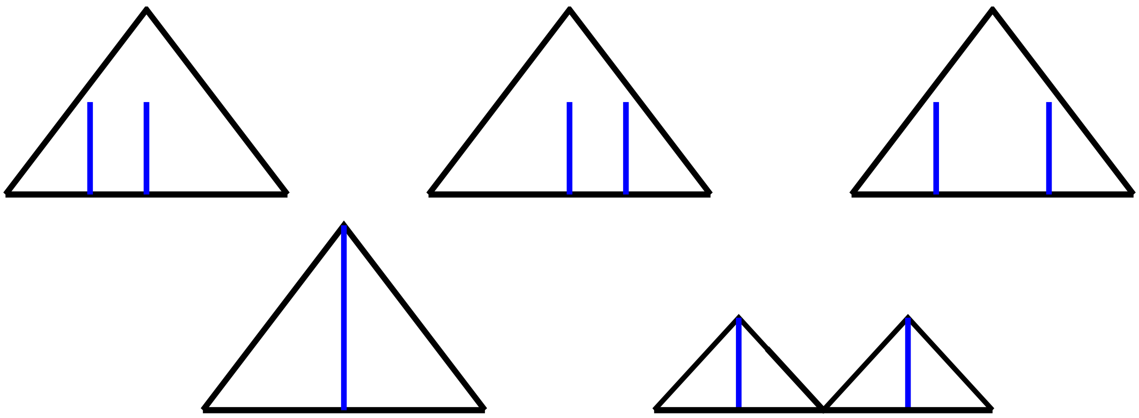

Figure 3.4: Example of a Dyck path with sum of column heights in this case $m = 2$ and half length $N = 2$.

We rewrite the generating functions defined above by forming partial sums over N by introducing the fixed column height partition functions

$$Z^{(w)}_m(t) = \sum_{N=0}^{\infty} a^{(w)}_{N,m} t^{2N} \quad \text{and} \quad Z'^{(w)}_m(t) = \sum_{N=0}^{\infty} b^{(w)}_{N,m} t^{2N}, \tag{3.2.3}$$

so that

$$G_w(t,q) = \sum_{m=0}^{\infty} Z_m^{(w)}(t) q^m \quad \text{and} \quad G'_w(t,q) = \sum_{m=0}^{\infty} Z_m'^{(w)}(t) q^m. \tag{3.2.4}$$

Alternatively, considering the fixed length partition functions

$$Q_N^{(w)}(q) = \sum_{m=0}^{\infty} a_{N,m}^{(w)} q^m, \quad \text{and} \quad Q_N'^{(w)}(q) = \sum_{m=0}^{\infty} b_{N,m}^{(w)} q^m, \tag{3.2.5}$$

we can write

$$G_w(t,q) = \sum_{N=0}^{\infty} Q_N^{(w)}(q) t^{2N}, \quad \text{and} \quad G'_w(t,q) = \sum_{N=0}^{\infty} Q_N'^{(w)}(q) t^{2N}. \tag{3.2.6}$$

Therefore $Q_N^{(w)}$ and $Q_N^{'(w)}$ are the coefficients of t^{2N} of the generating functions $G_w(t,q)$ and $G'_w(t,q)$, respectively. The case $m = 0$ reduces to the enumeration of Dyck paths without area weighting, and if we consider columns of maximal height for any Dyck path (corresponding to the largest value of m for which $Z_m^{(w)}(t)$ is non zero) then this is equivalent to counting area weighted Dyck paths [9].

In [7], the correspondence between generating functions and continued fractions has been discussed in detail. We start by developing the idea of writing a continued fraction of path diagram bounded by a Dyck path with all possibilities of sum of column heights under it. The continued fraction expansions (3.2.7) and (3.2.8) are simply the result of a direct geometrical correspondence [7, Theorem 3A], corresponding to the odd Euler numbers E_{2n+1}, which are the coefficients of a formal power series defined via the continued fraction

$$\cfrac{1}{1 - \cfrac{1.2t^2}{1 - \cfrac{2.3t^2}{\ddots}}}. \tag{3.2.7}$$

Similarly restricting the contact of path diagram at every southeast step of a Dyck path resulted in the following continued fraction expansion which is given in [7, Theorem 3B]

$$\cfrac{1}{1-\cfrac{1.1t^2}{1-\cfrac{2.2t^2}{\ddots}}}. \tag{3.2.8}$$

This idea was generalised in [24] by giving the possibilities of sum of column heights a weight of q. So the continued fraction expansion for $w \to \infty$ is given as

$$G(t,q) = \cfrac{1}{1-\cfrac{\alpha^2(1-q)(1-q^2)}{1-\cfrac{\alpha^2(1-q^2)(1-q^3)}{\ddots}}} \tag{3.2.9}$$

and

$$G'(t,q) = \cfrac{1}{1-\cfrac{\alpha^2(1-q)^2}{1-\cfrac{\alpha^2(1-q^2)^2}{\ddots}}}, \tag{3.2.10}$$

where

$$\alpha = \frac{t}{1-q}. \tag{3.2.11}$$

Below we shall use α and t interchangably, as convenient.

This enumeration is analogous to q-tangent and q-secant numbers, respectively.

Proposition 3.1. For $w \geq 0$, the generating functions in (3.2.1) and (3.2.2) represented as generalised finite continued fractions are respectively given by

$$G_w(t,q) = \cfrac{1}{1 - \cfrac{\alpha^2(1-q)(1-q^2)}{1 - \cfrac{\alpha^2(1-q^2)(1-q^3)}{\ddots - \cfrac{\alpha^2(1-q^{w-1})(1-q^w)}{1-\alpha^2(1-q^w)(1-q^{w+1})}}}} \tag{3.2.12}$$

and

$$G'_w(t,q) = \cfrac{1}{1 - \cfrac{\alpha^2(1-q)^2}{1 - \cfrac{\alpha^2(1-q^2)^2}{\ddots - \cfrac{\alpha^2(1-q^{w-1})^2}{1-\alpha^2(1-q^w)^2}}}}, \tag{3.2.13}$$

where $\alpha = \dfrac{t}{1-q}$.

Proof. The generalised continued fractions in (3.2.12) and (3.2.13) are the finite versions of equations (3) and (4) given by [11]. The Dyck path of height zero is the Dyck path with no step and so there is no possibility of columns under it. Hence $G_0(t,q) = 1$. From the combinatorial theory of continued fractions given by [7], if $X = (a_0, a_1, a_2, .., b_0, b_1, ..)$ the Stieltjes type continued fraction is

$$S_k(X,t) = \cfrac{1}{1 - \cfrac{a_0 b_1 t^2}{1 - \cfrac{a_1 b_2 t^2}{\ddots - \cfrac{a_{k-2} b_{k-1} t^2}{1 - a_{k-1} b_k t^2}}}}$$

where a_i corresponds to a northeast step starting at height i, b_j corresponds to a southeast step starting at height j, and t is conjugate to the length of the Dyck path. For the continued fraction (3.2.12) we see that $a_0 = 1$. For $k \geq 1$ we see that at height k the northeast step is starting at $k-1$ and so the possibility of column heights below a northeast step is $a_k = 1 + q + q^2 + \ldots + q^{k-1}$. Similarly at height k the southeast step starts at height k and so the possibility of column heights below a southeast step is $b_k = 1+q+q^2+q^3+\ldots+q^k$. For the continued fraction (3.2.13), the contact is restricted for every southeast step so we have the same possibility of column height as the northeast step, that is $a_k = b_k = 1 + q + q^2 + q^3 + \ldots + q^{k-1}$ for $k \geq 1$. □

It is obvious that we can write the generalised continued fractions as a rational function.

Proposition 3.2. For $w \geq 0$,

$$G_w(t,q) = \frac{P_w(\alpha,q)}{Q_w(\alpha,q)} \quad \text{and} \quad G'_w(t,q) = \frac{P'_w(\alpha,q)}{Q'_w(\alpha,q)}, \tag{3.2.14}$$

where

$$P_w = \begin{cases} 0 & w = -1 \\ 1 & w = 0 \\ P_{w-1} - \alpha^2(1-q^w)(1-q^{w+1})P_{w-2} & w \geq 1 \end{cases}, \tag{3.2.15}$$

$$Q_w = \begin{cases} 1 & w = -1 \\ 1 & w = 0 \\ Q_{w-1} - \alpha^2(1-q^w)(1-q^{w+1})Q_{w-2} & w \geq 1 \end{cases}, \tag{3.2.16}$$

$$P'_w = \begin{cases} 0 & w = -1 \\ 1 & w = 0 \\ P_{w-1} - \alpha^2(1-q^w)^2 P_{w-2} & w \geq 1 \end{cases} \quad \text{and} \tag{3.2.17}$$

$$Q'_w = \begin{cases} 1 & w = -1 \\ 1 & w = 0 \\ Q_{w-1} - \alpha^2(1-q^w)^2 Q_{w-2} & w \geq 1 \end{cases}. \tag{3.2.18}$$

Proof. The initial conditions follow from the fact that $G_{-1}(t,q) = G'_{-1}(t,q) = 0/1$. This implies that $P_{-1} = P'_{-1} = 0$ and $Q_{-1} = Q'_{-1} = 1$. Also for $w = 0$ we have $G_0(t,q) = G'_0(t,q) = 1/1$. For $w \geq 1$ we compare with the h-th convergent of the J-fraction on page 152 of [7]. We have $z = t$ and $a_k = 1$ for $k \geq 1$ and $b_k = (1-q^w)(1-q^{w+1})$ and $c_k = 0$ for $k \geq 0$. This reduces to the recurrence equations given in (3.2.15) and (3.2.16). For the generating function $G'_w(t,q)$ we see that, instead, $b_k = (1-q^w)(1-q^w)$, which results in the recurrence equations given in (3.2.17) and (3.2.18). $\square$

3.3 w^{th} convergent of q-tangent numbers

Theorem 3.3. *For $w \geq 0$, the w^{th} convergent for the continued fraction expansion of the generating function of q-tangent numbers is given by*

$$G_w(t,q) = \frac{1}{1 - \dfrac{\lambda^2(1-q)\left[\bar{\lambda}^w \phi(\lambda, q^3)\phi\left(\bar{\lambda}, q^{w+3}\right) - \lambda^w \phi\left(\bar{\lambda}, q^3\right)\phi(\lambda, q^{w+3})\right]}{(1+\lambda^2)\left[\bar{\lambda}^w \phi(\lambda, q^2)\phi\left(\bar{\lambda}, q^{w+3}\right) - \lambda^{w+2}\phi\left(\bar{\lambda}, q^2\right)\phi(\lambda, q^{w+3})\right]}}. \tag{3.3.1}$$

Here,

$$\phi(\lambda, x) = \sum_{k=0}^{\infty} \frac{(i\lambda; q)_k(-i\lambda; q)_k x^k}{(\lambda^2 q; q)_k (q;q)_k} = {}_2\phi_1(i\lambda, -i\lambda; \lambda^2 q; q, x), \tag{3.3.2}$$

where

$${}_2\phi_1(a, b; c; q, x) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k\, x^k}{(c;q)_k (q;q)_k}$$

is a basic hypergeometric function (q-hypergeometric function),

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$$

is the standard notation for the q-Pochhammer symbol, and λ is a root of $P(\lambda) = \lambda^2 - \lambda/\alpha + 1$ where $\alpha = t/(1-q)$, and $\bar{\lambda} = 1/\lambda$.

Proof. We shall prove Theorem 3.3 by solving the recurrence relation (3.2.15) and (3.2.16). We can write P_w and Q_w as the linear combination of two basic hypergeometric functions and determine the coefficients from the initial conditions of the recurrences given in Proposition 3.2.

We can see that for $w \geq 1$ the recurrence relation for $P_w(\alpha, q)$ and $Q_w(\alpha, q)$ is the same, so we represent them both by $R(w)$ and solve simultaneously. From the recursion given in (3.2.15) and (3.2.16) we have for $w \geq 1$,

$$R(w) = R(w-1) - \alpha^2(1-q^w)(1-q^{w+1})R(w-2). \tag{3.3.3}$$

Unlike a linear recurrence with constant coefficients, this cannot be solved by a standard method because we have w-dependent coefficients. Moreover, the occurrence of both q^w and q^{2w} poses a difficulty, so our next step will be to eliminate the term containing q^{2w} by appropriate rewriting of the recurrences. It is evident from the coefficient of $R(w-2)$ that multiplying by a q-factorial will simplify (3.3.3) appropriately. We thus rescale the recursion (3.3.3) as follows:

$$R(w) = \alpha^w (q;q)_{w+1} S(w). \tag{3.3.4}$$

This transformation of coefficients leads to the recurrence

$$\begin{aligned} \alpha^w (q;q)_{w+1} S(w) = \alpha^{w-1}&(q;q)_w S(w-1) \\ &- \alpha^2(1-q^w)(1-q^{w+1})\alpha^{w-2}(q;q)_{w-1} S(w-2), \end{aligned} \tag{3.3.5}$$

dividing by $\alpha^w (q;q)_{w+1}$ throughout gives

$$S(w) = \frac{S(w-1)}{\alpha(1-q^{w+1})} - S(w-2). \tag{3.3.6}$$

Rearranging terms leads to

$$S(w) - \frac{1}{\alpha} S(w-1) + S(w-2) = q^{w+1}(S(w) + S(w-2)) \tag{3.3.7}$$

for $w \geq 1$. This eliminates q^{2w} from the recurrence as intended, as the right hand side only contains a q^w prefactor. The left hand side of equation (3.3.7) is a linear homogeneous recurrence relation with a characteristic polynomial

$$P(\lambda) = \lambda^2 - \frac{\lambda}{\alpha} + 1. \tag{3.3.8}$$

The two roots λ_1 and λ_2 of the characteristic polynomial are reciprocal to each other,

$$\lambda_1 \lambda_2 = 1, \tag{3.3.9}$$

a fact that we will need to use below. The solution to the recurrence relation (3.3.7) can be written as a linear combination of the powers of the roots of the characteristic polynomial, if the right hand side of (3.3.7) was zero, and hence not q-dependent. To solve the recurrence (3.3.7) in general, we use the ansatz

$$S(w) = \lambda^w \sum_{k=0}^{\infty} c_k q^{kw}, \tag{3.3.10}$$

which has been shown to work when there are powers of q^w in such a linear recurrence [18] [19]. Substituting this ansatz into equation (3.3.7), we have

$$\begin{aligned}\lambda^w \sum_{k=0}^{\infty} c_k q^{kw} - \left(\frac{1}{\alpha}\right) \lambda^{w-1} \sum_{k=0}^{\infty} c_k q^{k(w-1)} + \lambda^{w-2} \sum_{k=0}^{\infty} c_k q^{k(w-2)} \\ = q^{w+1} \left(\lambda^n \sum_{k=0}^{\infty} c_k q^{kw} + \lambda^{w-2} \sum_{k=0}^{\infty} c_k q^{k(w-2)} \right).\end{aligned} \tag{3.3.11}$$

It turns out that we can manipulate this equation to arrive at a recurrence equation for the coefficients c_k. Dividing by λ^{w-2} throughout we find

$$\begin{aligned}\lambda^2 \sum_{k=0}^{\infty} c_k q^{kw} - \left(\frac{1}{\alpha}\right) \lambda \sum_{k=0}^{\infty} c_k q^{k(w-1)} + \sum_{k=0}^{\infty} c_k q^{k(w-2)} \\ = q^{w+1} (\lambda^2 \sum_{k=0}^{\infty} c_k q^{kw} + \sum_{k=0}^{\infty} c_k q^{k(w-2)}).\end{aligned} \tag{3.3.12}$$

We separate the terms containing c_0 on the left hand side and adjust the limits of summation throughout the equation,

$$\begin{aligned}(\lambda^2 - \frac{\lambda}{\alpha} + 1)c_0 + \sum_{k=1}^{\infty} q^{kw}(\lambda^2 - \frac{\lambda}{\alpha}q^{-k} + q^{-2k})c_k \\ = q^{w+1}\left(\lambda^2 \sum_{k=1}^{\infty} c_{k-1}q^{(k-1)w} + \sum_{k=1}^{\infty} c_{k-1}q^{(k-1)(w-2)}\right).\end{aligned} \tag{3.3.13}$$

To establish the recurrence relation we collect the terms with c_k and c_{k-1},

$$P(\lambda)c_0 + \sum_{k=1}^{\infty} q^{kw-2k}(\lambda^2 q^{2k} - \frac{\lambda}{\alpha}q^k + 1)c_k = \sum_{k=1}^{\infty}(\lambda^2 q^{kw+1} + q^{kw-2k+3})c_{k-1}. \tag{3.3.14}$$

From (3.3.8) we replace the coefficient of c_k with $P(\lambda q^k)$ to get,

$$P(\lambda)c_0 + \sum_{k=1}^{\infty} q^{kw-2k}P(\lambda q^k)c_k = \sum_{k=1}^{\infty} q^{kw-2k}(\lambda^2 q^{2k} + q^2)qc_{k-1}\ . \tag{3.3.15}$$

The recurrence relation for c_k can now be read off from

$$P(\lambda)c_0 + \sum_{k=1}^{\infty} q^{kw-2k}\left(P(\lambda q^k)c_k - (\lambda^2 q^{2k} + q^2)qc_{k-1}\right) = 0. \tag{3.3.16}$$

This equation is satisfied if $P(\lambda) = 0$ and all the coefficients in the sum vanish, i.e. $P(\lambda q^k)c_k - (\lambda^2 q^{2k} + q^2)qc_{k-1} = 0$. Now $P(\lambda) = 0$ implies

$$\lambda^2 - \frac{\lambda}{\alpha} + 1 = 0\,,$$

which equivalently can be used to express α in terms of λ as

$$\alpha = \frac{\lambda}{1+\lambda^2}\,. \tag{3.3.17}$$

Eliminating α in the characteristic polynomial (3.3.8), we factorise as

$$P(\mu) = (\mu - \lambda)(\mu - \frac{1}{\lambda}).$$

We need to evaluate $P(\lambda q^k)$, so we now let $\mu = \lambda q^k$ and simplify to get

$$P(\lambda q^k) = (1 - q^k)(1 - \lambda^2 q^k). \tag{3.3.18}$$

If $P(\lambda q^k)c_k - (\lambda^2 q^{2k} + q^2)qc_{k-1} = 0$ then for $k > 0$ and $c_0 = 1$ we have

$$c_k = \frac{(\lambda^2 q^{2k} + q^2)qc_{k-1}}{P(\lambda q^k)}. \tag{3.3.19}$$

Now substituting in the value of $P(\lambda q^k)$ from (3.3.18) in (3.3.19) and iterating it we have

$$c_k = \frac{(-\lambda^2; q^2)_k \, q^{3k}}{(q;q)_k(\lambda^2 q; q)_k}, \tag{3.3.20}$$

where we choose to write all products in terms of the q-Pochhammer symbol. The full solution to the recurrence equation (3.3.7) is a linear combination of the ansatz over both values of λ. Here $P(\lambda) = 0$ and also $P(\bar{\lambda}) = 0$. We can write the general solution for $S(w)$ as

$$S(w) = A\lambda^w \sum_{k=0}^{\infty} c_k(\lambda, q)q^{kw} + B\bar{\lambda}^w \sum_{k=0}^{\infty} c_k\left(\bar{\lambda}, q\right) q^{kw}. \tag{3.3.21}$$

We can now write the general solution in terms of a basic hypergeometric series. We define

$$\phi(\lambda, x) = \sum_{k=0}^{\infty} \frac{(-\lambda^2; q^2)_k x^k}{(q;q)_k(\lambda^2 q; q)_k} = \sum_{k=0}^{\infty} \frac{(i\lambda; q)_k(-i\lambda; q)_k x^k}{(\lambda^2 q; q)_k(q;q)_k} = {}_2\phi_1(i\lambda, -i\lambda; \lambda^2 q; q, x), \tag{3.3.22}$$

where

$$_2\phi_1(a,b;c;q,x) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k \, x^k}{(c;q)_k (q;q)_k}.$$

Using this notation, the general solution $S(w)$ can simply be written as

$$S(w) = A\lambda^w \phi(\lambda, q^{w+3}) + B\bar{\lambda}^w \phi\left(\bar{\lambda}, q^{w+3}\right). \tag{3.3.23}$$

Using the initial conditions we can solve for A and B. Our recursion for P_w and Q_w for $w \geq 1$ is similar, but with different initial conditions. First we solve for P_w with the following initial conditions

$$P_{-1} = R(-1) = 0 \qquad P_0 = R(0) = 1\,.$$

Since $R(w) = \alpha^w (q;q)_{w+1} S(w)$, we have

$$S(-1) = 0 \qquad S(0) = \frac{1}{1-q}\,. \tag{3.3.24}$$

We substitute the initial conditions (3.3.24) in equation (3.3.23) and solve for A and B. This gives two simultaneous equations as

$$\frac{1}{1-q} = A\phi\left(\lambda, q^3\right) + B\phi\left(\bar{\lambda}, q^3\right)\,,$$

$$0 = A\bar{\lambda}\phi(\lambda, q^2) + B\lambda\phi\left(\bar{\lambda}, q^2\right).$$

Solving these equations gives

$$A = \frac{-\lambda^2 \phi\left(\bar{\lambda}, q^2\right)}{(1-q)\left(\phi(\lambda,q^2)\phi\left(\bar{\lambda},q^3\right) - \lambda^2\phi\left(\bar{\lambda},q^2\right)\phi(\lambda,q^3)\right)}$$

and

$$B = \frac{\phi(\lambda, q^2)}{(1-q)\left(\phi(\lambda,q^2)\phi\left(\bar{\lambda},q^3\right) - \lambda^2\phi\left(\bar{\lambda},q^2\right)\phi(\lambda,q^3)\right)}\,.$$

Similarly we solve for Q_w. Let the general solution be

$$S(w) = C\lambda^n \phi(\lambda, q^{w+3}) + D\bar{\lambda}^w \phi\left(\bar{\lambda}, q^{w+3}\right), \tag{3.3.25}$$

with the initial conditions as

$$R(-1) = 1 \qquad R(0) = 1.$$

Since $R(w) = \alpha^w (q;q)_{w+1} S(w)$ we have

$$S(-1) = \alpha \qquad S(0) = \frac{1}{1-q}. \tag{3.3.26}$$

Substituting the initial condition from (3.3.26) in equation (3.3.25), we get two equations

$$\begin{aligned} \frac{1}{1-q} &= C\phi(\lambda, q^3) + D\phi\left(\bar{\lambda}, q^3\right) \\ \alpha &= C\bar{\lambda}\phi(\lambda, q^2) + D\lambda\phi\left(\bar{\lambda}, q^2\right) \ , \end{aligned}$$

solving for C and D gives

$$C = \frac{(\alpha)(\lambda)(1-q)\phi\left(\bar{\lambda}, q^3\right) - \lambda^2 \phi\left(\bar{\lambda}, q^2\right)}{(1-q)\left(\phi(\lambda, q^2)\phi\left(\bar{\lambda}, q^3\right) - \lambda^2 \phi\left(\bar{\lambda}, q^2\right)\phi(\lambda, q^3)\right)}$$

and

$$D = \frac{\phi(\lambda, q^2) - (\alpha)(\lambda)(1-q)\phi(\lambda, q^3)}{(1-q)\left(\phi(\lambda, q^2)\phi\left(\bar{\lambda}, q^3\right) - \lambda^2 \phi\left(\bar{\lambda}, q^2\right)\phi(\lambda, q^3)\right)}.$$

Substituting the full solution for $P_w(\alpha, q)$ and $Q_w(\alpha, q)$ in (3.2.14), we have the following expression

$$G_w(t, q) = \frac{A\lambda^w \phi(\lambda, q^{w+3}) + B(\bar{\lambda})^w \phi\left(\bar{\lambda}, q^{w+3}\right)}{C\lambda^w \phi(\lambda, q^{w+3}) + D(\bar{\lambda})^w \phi\left(\bar{\lambda}, q^{w+3}\right)}. \tag{3.3.27}$$

Using the values of A, B, C and D we have the full solution given by (3.3.1). $\square$

3.4 Half plane limit for q-tangent numbers

By taking the limit of infinite w in the generating function G_w, we derive an expression for the generating function of q-tangent numbers. This corresponds to enumeration of path diagrams in the half plane without height restriction, and we therefore refer to this as the half plane limit.

Corollary 3.4. The generating function of q-tangent numbers is

$$G(t,q) = \frac{(1+\lambda)^2 \left[1 - (1+\lambda^2) \sum_{k=0}^{\infty} \frac{(-i\lambda)^k}{(1-i\lambda q^k)}\right]}{\lambda^2(1-q)}, \tag{3.4.1}$$

where λ is the root of $t = (1-q)\lambda/(1+\lambda^2)$ with smallest modulus.

Proof. For the half plane limit we consider the sum (3.3.1). We know that the q series converges when $|q| < 1$ using the ratio test. From (3.3.9) we see that one of the roots of the characteristic polynomial (3.3.8) is less than one if t is sufficiently small. We assume $|\lambda| < 1$. When $w \to \infty$,

$$\phi(\lambda, q^{w+3}) = {}_2\phi_1(i\lambda, -i\lambda; \lambda^2 q; q, q^{w+3}) \to {}_2\phi_1(i\lambda, -i\lambda; \lambda^2 q; q, 0) = 1.$$

Also

$$|\lambda^w| \to 0.$$

This implies

$$G(t,q) = \frac{1}{1 - \dfrac{\lambda^2(1-q)\phi(\lambda, q^3)}{(1+\lambda^2)\phi(\lambda, q^2)}}. \tag{3.4.2}$$

Heine's transformation formula for ${}_2\phi_1$ series [8] is given by

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\, {}_2\phi_1(c/b,z;az;q,b). \tag{3.4.3}$$

Using this transformation we can write the basic hypergeometric functions in (3.4.2) as follows

$$\phi(\lambda,q^2) = {}_2\phi_1(i\lambda,-i\lambda;\lambda^2 q;q,q^2) = \frac{(-i\lambda;q)_\infty(i\lambda q^2;q)_\infty}{(\lambda^2 q;q)_\infty(q^2;q)_\infty}\, {}_2\phi_1(i\lambda q,q^2;i\lambda q^2;q,-i\lambda) \tag{3.4.4}$$

and

$$\phi(\lambda,q^3) = {}_2\phi_1(i\lambda,-i\lambda;\lambda^2 q;q,q^3) = \frac{(-i\lambda;q)_\infty(i\lambda q^3;q)_\infty}{(\lambda^2 q;q)_\infty(q^3;q)_\infty}\, {}_2\phi_1(i\lambda q,q^3;i\lambda q^3;q,-i\lambda). \tag{3.4.5}$$

Further substituting the transformations of basic hypergeometric functions from (3.4.4) and (3.4.5) in the half plane limit (3.4.2) yields

$$G(t,q) = \frac{1}{1-\dfrac{\lambda^2(1-q)\dfrac{(-i\lambda;q)_\infty(i\lambda q^3;q)_\infty}{(\lambda^2 q;q)_\infty(q^3;q)_\infty}\, {}_2\phi_1(i\lambda q,q^3;i\lambda q^3;q,-i\lambda)}{(1+\lambda^2)\dfrac{(-i\lambda;q)_\infty(i\lambda q^2;q)_\infty}{(\lambda^2 q;q)_\infty(q^2;q)_\infty}\, {}_2\phi_1(i\lambda q,q^2;i\lambda q^2;q,-i\lambda)}}. \tag{3.4.6}$$

Simplifying this expression gives

$$G(t,q) = \cfrac{1}{1 - \cfrac{\lambda^2(1-q)(1-q^2)\sum_{k=0}^{\infty}\frac{(i\lambda q;q)_k(q^3;q)_k}{(i\lambda q^3;q)_k(q;q)_k}(-i\lambda)^k}{(1-i\lambda q^2)(1+\lambda^2)\sum_{k=0}^{\infty}\frac{(i\lambda q;q)_k(q^2;q)_k}{(i\lambda q^2;q)_k(q;q)_k}(-i\lambda)^k}} \tag{3.4.7}$$

$$= \cfrac{1}{1 - \cfrac{(\lambda^2)(1-q)\sum_{k=0}^{\infty}\frac{(1-q^{k+1})(1-q^{k+2})}{(1-i\lambda q^{k+1})(1-i\lambda q^{k+2})}(-i\lambda)^k}{(1+\lambda^2)\sum_{k=0}^{\infty}\frac{(1-q^{k+1})}{(1-i\lambda q^{k+1})}(-i\lambda)^k}}. \tag{3.4.8}$$

We aim to simplify the terms in the sums on the right hand side of (3.4.8). For this we let

$$N = \frac{(1-q^{k+1})(1-q^{k+2})}{(1-i\lambda q^{k+1})(1-i\lambda q^{k+2})}(-i\lambda)^k \tag{3.4.9}$$

and

$$D = \frac{(1-q^{k+1})}{(1-i\lambda q^{k+1})}(-i\lambda)^k. \tag{3.4.10}$$

To simplify, we substitute $x = -i\lambda$ in N to get

$$N = \sum_{k=0}^{\infty}\frac{(1-q^{k+1})(1-q^{k+2})}{(1+xq^{k+1})(1+xq^{k+2})}x^k. \tag{3.4.11}$$

Using partial fraction decomposition we get

$$N = \sum_{k=0}^{\infty}\left[\frac{x^k}{x^2} - \frac{x^k(1+x)(x+q)}{x^2(q-1)(1+xq^{k+1})} + \frac{x^k(1+x)(1+xq)}{x^2(q-1)(1+xq^{k+2})}\right]. \tag{3.4.12}$$

Shifting summation indices and combining the fractions, we find

$$N = (-1)\frac{x^4 \sum\limits_{k=0}^{\infty} \frac{x^k}{(1+xq^{k+1})} - 2x^2 \sum\limits_{k=0}^{\infty} \frac{x^k}{(1+xq^{k+1})} + \sum\limits_{k=0}^{\infty} \frac{x^k}{(1+xq^{k+1})} + x^2 q + 1}{x^4(q-1)(x-1)}. \tag{3.4.13}$$

Similarly we consider the expression D and substitute $x = -i\lambda$ to get

$$D = \sum_{k=0}^{\infty} \frac{(1-q^{k+1})}{(1+xq^{k+1})} x^k.$$

Again for this expression we decompose into partial fractions

$$D = \sum_{k=0}^{\infty} -\frac{x^k}{x} + \frac{x^k(x+1)}{x(1+xq^{k+1})}. \tag{3.4.14}$$

Shifting summation indices and combining the fractions, we find

$$D = \frac{x^2 \sum\limits_{k=0}^{\infty} \frac{x^k}{(1+xq^k)} - \sum\limits_{k=0}^{\infty} \frac{x^k}{(1+xq^k)} + 1}{x^2(x-1)}. \tag{3.4.15}$$

Substituting (3.4.13) and (3.4.15) in (3.4.8) and simplifying, we get the final expression (3.4.1). □

3.4.1 An explicit formula for q-tangent numbers

Next we make connections to the previous work in [11] where closed formulas for q-Euler numbers (q-tangent numbers and q-secant numbers) are obtained. The paper [11] uses permutation tableaux for obtaining these formulas. In particular, we extract the coefficient of t^{2N} of $G(t, q)$ given in (3.4.1). Our result is equivalent to Theorem 1.4 in [11]. We use the result in [11] and relabel $k = m$, $n = N$, $i = l + m$ and simplify which results in the expression (3.4.16).

Corollary 3.5. $Q_N(q) = [t^{2N}]G(t,q)$ is given by

$$Q_N(q) = \frac{1}{(1-q)^{2N+1}} \sum_{m=0}^{N} \frac{q^{m^2+2m} \left(\sum_{l=-m}^{m+1} (-1)^l q^{-l^2+2l} \right) (2m+2) \binom{2N+1}{N+m+1}}{N+m+2} \tag{3.4.16}$$

To prove this corollary, we need an identity which can be obtained from counting rectangles on the square lattice in two different ways, taking ideas from [21].

Lemma 3.6.

$$\sum_{n=0}^{\infty} \frac{x^n}{1-yq^n} = \sum_{n=0}^{\infty} \frac{x^n y^n q^{n^2}(1-xyq^{2n})}{(1-xq^n)(1-yq^n)} \ . \tag{3.4.17}$$

Proof. We consider the generating function of rectangles (including those of height or width zero) on the square lattice, counted with respect to height, width, and area, given by

$$R(x,y,q) = \sum_{n,m=0}^{\infty} x^n y^m q^{nm} \ . \tag{3.4.18}$$

Summing over m gives the left hand side of identity (3.4.17). If we instead sum over rectangles of fixed minimal width or height N, then this results in

$$R(x,y,q) = \sum_{N=0}^{\infty} \left(\sum_{m=N}^{\infty} x^N y^m q^{Nm} + \sum_{n=N}^{\infty} x^n y^N q^{nN} - x^N y^N q^{N^2} \right) \tag{3.4.19}$$

$$= \sum_{N=0}^{\infty} \left(\frac{x^N y^N q^{N^2}}{1-yq^N} + \frac{x^N y^N q^{N^2}}{1-xq^N} - x^N y^N q^{N^2} \right) \tag{3.4.20}$$

$$= \sum_{N=0}^{\infty} x^N y^N q^{N^2} \left(\frac{1}{1-yq^N} + \frac{1}{1-xq^N} - 1 \right) \ . \tag{3.4.21}$$

Simplifying the terms in the final bracket gives the right hand side of identity (3.4.17). □

Proof of Corollary 3.5. The sum in (3.4.1) can be identified with $R(-i\lambda, i\lambda, q)$, so that using Lemma 3.6 we get

$$G(t,q) = \frac{(1+\lambda)^2 \left[1 - (1+\lambda^2) \sum_{n=0}^{\infty} \frac{q^{n^2} \lambda^{2n} (1-\lambda^2 q^{2n})}{(1+\lambda^2 q^{2n})} \right]}{\lambda^2 (1-q)} . \tag{3.4.22}$$

We remind that $G(t,q)$ is by definition an even function in t, and that the t-dependence on the right hand side is implicit in $\lambda = \lambda(t)$. To extract the coefficient of t^{2N}, we evaluate the contour integral

$$[t^{2N}]G(t,q) = \frac{1}{2\pi i} \oint \frac{(1+\lambda(t)^2) \left(1 - (1+\lambda(t))^2 \sum_{n=0}^{\infty} \frac{q^{n^2} \lambda(t)^{2n} (1-\lambda(t)^2 q^{2n})}{(1+\lambda(t)^2 q^{2n})} \right)}{\lambda(t)^2 (1-q) t^{2N+1}} dt . \tag{3.4.23}$$

Next we perform the change of variables from t to λ. Comparing the value of α from the characteristic polynomial given in (3.3.8) and (3.2.11), we have

$$\lambda + \frac{1}{\lambda} = \frac{1-q}{t}, \tag{3.4.24}$$

this implies

$$t = \frac{(1-q)\lambda}{1+\lambda^2}. \tag{3.4.25}$$

Differentiating t given in (3.4.25) with respect to λ gives

$$\frac{dt}{d\lambda} = \frac{(1-q)(1-\lambda)(1+\lambda)}{(1+\lambda^2)^2}. \tag{3.4.26}$$

Substituting the value of t and its derivative from (3.4.25) and (3.4.26) in (3.4.23) we get

$$[t^{2N}]G(t,q) = \frac{1}{2\pi i} \oint \left(\frac{(1+\lambda^2)\left(1-(1+\lambda^2)\sum_{n=0}^{\infty} \frac{q^{n^2}\lambda^{2n}(1-\lambda^2 q^{2n})}{(1+\lambda^2 q^{2n})}\right)(1-q)(1+\lambda^2)^{2N+1}(1-\lambda^2)}{\lambda^{2N+2}(1-q)(1-q)^{2N+1}(1+\lambda^2)^2} \right) \frac{d\lambda}{\lambda}. \tag{3.4.27}$$

Simplifying gives

$$[t^{2N}]G(t,q) = \frac{1}{2\pi i} \oint \left(\frac{(1+\lambda^2)^{2N}\left(1-(1+\lambda^2)\sum_{n=0}^{\infty} \frac{q^{n^2}\lambda^{2n}(1-\lambda^2 q^{2n})}{(1+\lambda^2 q^{2n})}\right)(1-\lambda^2)}{\lambda^{2N+2}(1-q)^{2N+1}} \right) \frac{d\lambda}{\lambda}. \tag{3.4.28}$$

The expression above is in terms of λ, and using the relationship between λ and t we know

$$[t^{2N}]G\left(t,q\right) = [\lambda^0]H_N(\lambda,q), \tag{3.4.29}$$

where

$$H_N(\lambda,q) = \frac{(1+\lambda^2)^{2N}\left(1-(1+\lambda^2)\sum_{n=0}^{\infty} \frac{q^{n^2}\lambda^{2n}(1-\lambda^2 q^{2n})}{(1+\lambda^2 q^{2n})}\right)(1-\lambda^2)}{\lambda^{2N+2}(1-q)^{2N+1}} \tag{3.4.30}$$

$$= \frac{\left(\lambda+\frac{1}{\lambda}\right)^{2N}\left(\frac{1}{\lambda^2}-1\right)\left(1-(1+\lambda^2)\sum_{n=0}^{\infty} \frac{q^{n^2}\lambda^{2n}(1-\lambda^2 q^{2n})}{(1+\lambda^2 q^{2n})}\right)}{(1-q)^{2N+1}}. \tag{3.4.31}$$

We are therefore led to computing the constant term in λ of $H_N(\lambda, q)$, i.e.

$$[\lambda^0]H_N(\lambda, q) = [\lambda^0]\frac{T_1 - T_2 \sum_{n=0}^{\infty} T_3}{((1-q)^{2N+1}} \ , \tag{3.4.32}$$

where

$$T_1 = \left(\lambda + \frac{1}{\lambda}\right)^{2N} \left(\frac{1}{\lambda^2} - 1\right),$$

$$T_2 = \left(\lambda + \frac{1}{\lambda}\right)^{2N} \left(\frac{1}{\lambda^2} - 1\right)(1 + \lambda^2),$$

and

$$T_3 = \frac{q^{n^2}\lambda^{2n}(1 - \lambda^2 q^{2n})}{(1 + \lambda^2 q^{2n})}.$$

We can write these T_1, T_2 and T_3 as

$$T_1 = \sum_{k=0}^{2N+1} \frac{(2N)!(2N - 2k + 1)\lambda^{2k-2N-2}}{k!(2N - k + 1)!} \ , \tag{3.4.33}$$

$$T_2 = \sum_{k=0}^{2N+2} \frac{(2N+1)!(2N - 2k + 2)\lambda^{2k-2N-2}}{k!(2N - k + 2)!} \tag{3.4.34}$$

and

$$T_3 = q^{n^2}\lambda^{2n}\left(2\left(\sum_{l=0}^{\infty}(-1)^l(\lambda^2 q^{2n})^l\right) - 1\right). \tag{3.4.35}$$

Substituting the expression (3.4.33),(3.4.34) and (3.4.35) in (3.4.32) gives

$$H_N(\lambda, q) = \frac{1}{(1-q)^{2N+1}}\left(\sum_{k=0}^{2N+1} \frac{(2N)!(2N - 2k + 1)\lambda^{2k-2N-2}}{k!(2N - k + 1)!} - \right.$$
$$\left.\left(\sum_{k=0}^{2N+2} \frac{(2N+1)!(2N - 2k + 2)\lambda^{2k-2N-2}}{k!(2N - k + 2)!}\right)\left(\sum_{n=0}^{\infty} q^{n^2}\lambda^{2n}\left(2\sum_{l=0}^{\infty}(-1)^l(\lambda^2 q^{2n})^l - 1\right)\right)\right). \tag{3.4.36}$$

Further we simplify by expanding and combining the summations to get

$$
\begin{aligned}
H_N(\lambda, q) = \frac{1}{(1-q)^{2N+1}} \Bigg(& \sum_{k=0}^{2N+1} \frac{(2N)!(2N-2k+1)\lambda^{2k-2N-2}}{k!(2N-k+1)!} \\
& -2 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{2N+2} (-1)^l q^{n^2+2nl} \frac{(2N+1)!(2N-2k+2)\lambda^{2k-2N-2+2n+2l}}{k!(2N-k+2)!} \\
& + \sum_{n=0}^{\infty} \sum_{k=0}^{2N+2} q^{n^2} \frac{(2N+1)!(2N-2k+2)\lambda^{2k-2N-2+2n}}{k!(2N-k+2+2n)!} \Bigg) . \qquad (3.4.37)
\end{aligned}
$$

We want to extract the coefficient of λ^0, so we combine the powers of λ and equate them to 0 and find k. We insert this value of k into each term to get the coefficient of λ^0. For the first term in (3.4.37) we equate $2k - 2N - 2 = 0$ and get $k = N + 1$. For the second term we equate $2k - 2N - 2 + 2n + 2l = 0$ and get $k = N - n - l + 1$. Finally for the third term we equate $2k - 2N - 2 + 2n = 0$ and get $k = N - n + 1$. Inserting these values of k in their respective terms we get

$$
\begin{aligned}
Q_N(q) = \frac{1}{(1-q)^{2N+1}} \Bigg(& -\frac{(2N)!}{N!(N+1)!} \\
& -2 \sum_{n=0}^{N+1} \sum_{l=0}^{N-n+1} (-1)^l q^{n^2+2nl} \frac{(2N+1)!(2n+2l)}{(N-n-l+1)!(N+n+l+1)!} \\
& + \sum_{n=0}^{N+1} q^{n^2} \frac{(2N+1)!(2n)}{(N-n+1)!(N+n+1)!} \Bigg) . \qquad (3.4.38)
\end{aligned}
$$

Next we add and subtract the term $-l^2$ to the power of q in second term to complete the square

$$Q_N(q) = \frac{1}{(1-q)^{2N+1}} \Bigg(-\frac{(2N)!}{N!(N+1)!} \\ -2\sum_{n=0}^{N+1}\sum_{l=0}^{N-n+1} (-1)^l q^{(n+l)^2} q^{-l^2} \frac{(2N+1)!(2n+2l)}{(N-n-l+1)!(N+n+l+1)!} \\ +\sum_{n=0}^{N+1} q^{n^2} \frac{(2N+1)!(2n)}{(N-n+1)!(N+n+1)!} \Bigg). \quad (3.4.39)$$

The sum runs over $n+l \leq N$, so we substitute $n+l=m$ in the second term which simplifies the expression as

$$Q_N(q) = \frac{1}{(1-q)^{2N+1}} \Bigg(-\frac{(2N)!}{N!(N+1)!} \\ -2\sum_{m=0}^{N+1}\sum_{l=0}^{m} (-1)^l q^{m^2} q^{-l^2} \frac{(2N+1)!(2m)}{(N-m+1)!(N+m+1)!} + \sum_{n=0}^{N+1} q^{n^2} \frac{(2N+1)!(2n)}{(N-n+1)!(N+n+1)!} \Bigg). \quad (3.4.40)$$

Next we change the variable of summation from n to m in the third term and simplify to get

$$Q_N(q) = \frac{1}{(1-q)^{2N+1}} \Bigg(-\frac{(2N)!}{N!(N+1)!} \\ -\sum_{m=0}^{N+1} q^{m^2} \frac{(2N+1)!(2m)}{(N-m+1)!(N+m+1)!} \left(2\sum_{l=0}^{m+1} (-1)^l q^{-l^2} - 1 \right) \Bigg). \quad (3.4.41)$$

Extending the limits of l in the third term gives,

$$Q_N(q) = \frac{1}{(1-q)^{2N+1}} \Bigg(-\frac{(2N)!}{N!(N+1)!} \\ -\sum_{m=0}^{N+1} q^{m^2} \frac{(2N+1)!(2m)}{(N-m+1)!(N+m+1)!} \left(\sum_{l=-(m+1)}^{m+1} (-1)^l q^{-l^2} \right) \Bigg). \quad (3.4.42)$$

Replacing m by $m+1$, and l by $l-1$ gives

$$Q_N(q) = \frac{1}{(1-q)^{2N+1}} \Bigg(-\frac{(2N)!}{N!(N+1)!} + \sum_{m=0}^{N} q^{m^2+2m} \frac{(2N+1)!(2m+2)}{(N-m)!(N+m+2)!} \left(\sum_{l=-m}^{m+2} (-1)^l q^{-l^2+2l} \right) \Bigg). \quad (3.4.43)$$

Now in the second term $l = m+2$ gives

$$\frac{(2N)!}{N!(N+1)!}.$$

This cancels out with the first term and we get the final expression as (3.4.16). $\square$

3.5 w^{th} convergent of q-secant numbers

Theorem 3.7. *For $w \geq 0$, the w^{th} convergent for the continued fraction expansion of the generating function of q-secant numbers is given by*

$$G'_w(t,q) = \frac{1}{1 - \dfrac{\lambda^2(1-q)\left[\bar{\lambda}^w \psi(\lambda, q^2)\psi\left(\bar{\lambda}, q^{w+2}\right) - \lambda^w \psi\left(\bar{\lambda}, q^2\right)\psi(\lambda, q^{w+2})\right]}{(1+\lambda^2)\left[\bar{\lambda}^w \psi(\lambda, q)\psi\left(\bar{\lambda}, q^{w+2}\right) - \lambda^{w+2}\psi\left(\bar{\lambda}, q\right)\psi(\lambda, q^{w+2})\right]}}. \quad (3.5.1)$$

Here,

$$\psi(\lambda, x) = \sum_{k=0}^{\infty} \frac{(i\lambda\sqrt{q}; q)_k(-i\lambda\sqrt{q}; q)_k x^k}{(\lambda^2 q; q)_k (q;q)_k} = {}_2\phi_1(i\lambda\sqrt{q}, -i\lambda\sqrt{q}; \lambda^2 q; q, x),$$

where

$${}_2\phi_1(a, b; c; q, x) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k \, x^k}{(c;q)_k (q;q)_k}$$

is a basic hypergeometric function (q-hypergeometric function) and λ *is the root of* $P(\lambda) = \lambda^2 - \lambda/\alpha + 1$ *where* $\alpha = t/(1-q)$ *and* $\bar{\lambda} = 1/\lambda$.

Proof. We shall prove Theorem 3.7 by solving the recurrence (3.2.17) and (3.2.18). This is done as in Theorem 3.3. It follows from the continued fraction expansion given in (3.2.13) that both the numerator $P'_w(\alpha, q)$ and denominator $Q'_w(\alpha, q)$ satisfy the recurrence relations given in (3.2.17) and (3.2.18) respectively. As the recursions are same for $w \geq 1$, therefore we represent them both by $R(w)$ and solve simultaneously. It follows that

$$R(w) = R(w-1) - \alpha^2(1-q^w)^2 R(w-2). \tag{3.5.2}$$

Expanding the coefficient of $R(w-2)$ gives three terms which cannot be solved explicitly using standard method because we have w-dependent coefficients. Also the terms q^w and q^{2w} create difficulty, so we will aim to eliminate the terms containing q^{2w} by suitable rescaling. For this we use the ansatz (3.3.4). This transformation of coefficients leads to

$$\begin{aligned}\alpha^w (q;q)_{w+1} S(w) = \alpha^{w-1}&(q;q)_w S(w-1)\\ &- \alpha^2(1-q^w)(1-q^w)\alpha^{w-2}(q;q)_{w-1} S(w-2).\end{aligned} \tag{3.5.3}$$

Dividing by $\alpha^w(q;q)_w$, we obtain

$$(1-q^{w+1})S(w) = \frac{1}{\alpha} S(w-1) - (1-q^w)S(w-2). \tag{3.5.4}$$

Rearranging to get the linear recurrence with constant coefficients on the left hand side

$$S(w) - \frac{1}{\alpha}S(w-1) + S(w-2) = q^{w+1}S(w) + q^w S(w-2) \tag{3.5.5}$$

for $w \geq 1$. This eliminates the q^{2w} from the recurrence as intended, with only q^w factors on the right hand side. We see that this recurrence is very similar to (3.3.7). The left hand side of (3.5.5) is a linear homogeneous recurrence relation with the same characteristic polynomial (3.3.8) as above, however the right hand side is slightly different, with a prefactor of q^w in front of $S(w-2)$ instead of a prefactor q^{w+1}. We thus use the same ansatz (3.3.10) to solve the recurrence. Following a calculation identical to the one for q-tangent numbers, we find for $k > 0$

$$c_k = \frac{(\lambda^2 q^{2k} + q) q c_{k-1}}{P(\lambda q^k)}. \tag{3.5.6}$$

Now substituting the value of $P(\lambda q^k)$ in (3.5.6) and iterating it, we get

$$c_k = \frac{(-\lambda^2 q; q^2)_k \, q^{2k}}{(q;q)_k (\lambda^2 q; q)_k} \,. \tag{3.5.7}$$

The full solution to the recurrence equation (3.5.5) is a linear combination of the ansatz over both the values of λ. Here $P(\lambda) = 0$ and also $P(\bar{\lambda}) = 0$ (where $\bar{\lambda} = \frac{1}{\lambda}$). We can write the general solution for $S(w)$ as

$$S(w) = A\lambda^w \sum_{k=0}^{\infty} c_k(\lambda, q) q^{kw} + B\bar{\lambda}^w \sum_{k=0}^{\infty} c_k\left(\bar{\lambda}, q\right) q^{kw}. \tag{3.5.8}$$

We define

$$\begin{aligned} \psi(\lambda, x) &= \sum_{k=0}^{\infty} \frac{(-\lambda^2 q; q^2)_k \, x^k}{(q;q)_k (\lambda^2 q; q)_k} = \sum_{k=0}^{\infty} \frac{(i\lambda\sqrt{q}; q)_k (-i\lambda\sqrt{q}; q)_k x^k}{(\lambda^2 q; q)_k (q;q)_k} \\ &= {}_2\phi_1(i\lambda\sqrt{q}, -i\lambda\sqrt{q}; \lambda^2 q; q, x) \end{aligned}$$

where ${}_2\phi_1$ is a basic hypergeometric function. The general solution can be expressed as follows

$$S(w) = A\lambda^n \psi(\lambda, q^{w+2}) + B\bar{\lambda}^w \psi\left(\bar{\lambda}, q^{w+2}\right). \tag{3.5.9}$$

Using the initial conditions, we can solve for A and B. Our recurrence relation was same for P'_w and Q'_w for $w \geq 1$, but the initial conditions were different as given in (3.2.17) and (3.2.18). First solving it for the P'_w with the initial conditions as

$$P'_{-1} = R(-1) = 0 \qquad P'_0 = R(0) = 1 \; .$$

Since $R(w) = \alpha^w (q;q)_{w+1} S(w)$, we have

$$S(-1) = 0 \qquad S(0) = \frac{1}{1-q} \; .$$

Substituting in equation (3.5.9) and solving it for A and B, we get two simultaneous equations

$$\begin{aligned} \frac{1}{1-q} &= A\psi\left(\lambda, q^2\right) + B\psi\left(\bar{\lambda}, q^2\right), \\ 0 &= A\bar{\lambda}\psi(\lambda, q) + B\lambda\psi\left(\bar{\lambda}, q\right). \end{aligned}$$

Solving these equations gives

$$A = \frac{-\lambda^2\psi\left(\bar{\lambda}, q\right)}{(1-q)\left(\psi(\lambda,q)\psi\left(\bar{\lambda},q^2\right) - \lambda^2\psi\left(\bar{\lambda},q\right)\psi(\lambda,q^2)\right)}$$

and

$$B = \frac{\psi(\lambda, q)}{(1-q)\left(\psi(\lambda,q)\psi\left(\bar{\lambda},q^2\right) - \lambda^2\psi\left(\bar{\lambda},q\right)\psi(\lambda,q^2)\right)}.$$

Similarly, we solve for Q'_w. We let our general solution be

$$S(w) = C\lambda^w\psi(\lambda, q^{w+2}) + D\bar{\lambda}^w\psi\left(\bar{\lambda}, q^{w+2}\right). \tag{3.5.10}$$

The initial conditions are

$$Q'_{-1} = R(-1) = 1 \qquad Q'_0 = R(0) = 1 \; .$$

Since $R(w) = \alpha^w (q;q)_{w+1} S(w)$, we have

$$S(-1) = \alpha \qquad \text{and} \qquad S(0) = \frac{1}{1-q}.$$

Substituting in equation (3.5.10) we get two equations

$$\frac{1}{1-q} = C\psi(\lambda, q^2) + D\psi\left(\bar{\lambda}, q^2\right),$$

$$\alpha = C\bar{\lambda}\psi(\lambda, q) + D\lambda\psi\left(\bar{\lambda}, q\right).$$

Solving for C and D, we obtain

$$C = \frac{(\alpha)(\lambda)(1-q)\psi\left(\bar{\lambda}, q^2\right) - \lambda^2\psi\left(\bar{\lambda}, q\right)}{(1-q)\left(\psi(\lambda, q)\psi\left(\bar{\lambda}, q^2\right) - \lambda^2\psi\left(\bar{\lambda}, q\right)\psi(\lambda, q^2)\right)}$$

and

$$D = \frac{\psi(\lambda, q) - (\alpha)(\lambda)(1-q)\psi(\lambda, q^2)}{(1-q)\left(\psi(\lambda, q)\psi\left(\bar{\lambda}, q^2\right) - \lambda^2\psi\left(\bar{\lambda}, q\right)\psi(\lambda, q^2)\right)}.$$

Substituting the full solution for $P'_w(\alpha, q)$ and $Q'_w(\alpha, q)$ in (3.2.14), we have the generating function as

$$G'_w(t,q) = \frac{A\lambda^n\psi(\lambda, q^{w+2}) + B(\bar{\lambda})^w\psi\left(\bar{\lambda}, q^{w+2}\right)}{C\lambda^w\psi(\lambda, q^{w+2}) + D(\bar{\lambda})^w\psi\left(\bar{\lambda}, q^{w+2}\right)}. \tag{3.5.11}$$

Using the values of A, B, C and D, the full solution is given by (3.5.1). $\square$

3.6 Half plane limit for q-secant numbers

By taking the limit of infinite w in the generating function G'_w, we derive an expression for the generating function of q-secant numbers. We refer to this as the half plane limit as it corresponds to the enumeration of path diagrams without any height restriction.

Corollary 3.8. The generating function of q-secant numbers is

$$G'(t,q) = (1+\lambda^2)\sum_{k=0}^{\infty} \frac{(-i\lambda\sqrt{q})^k}{(1-i\lambda\sqrt{q}q^k)}\ , \tag{3.6.1}$$

where λ is the root of $t(1-q)\lambda/(1+\lambda^2)$ with smallest modulus.

Proof. For the half plane limit of q-secant numbers, consider the sum (3.5.1). We know that the q series converges when $|q|<1$ using the ratio test. From (3.3.9) we see that one of the roots of characteristic polynomial (3.3.8) is less than one if t is sufficiently small. Assume $|\lambda|<1$. For $w\to\infty$ we have

$$\psi(\lambda, q^{w+2}) = {}_2\phi_1(i\lambda\sqrt{q}, -i\lambda\sqrt{q}; \lambda^2 q; q, q^{w+2}) \to {}_2\phi_1(i\lambda\sqrt{q}, -i\lambda\sqrt{q}; \lambda^2 q; q, 0) = 1$$

and

$$|\lambda^w| \to 0.$$

This implies

$$G'(t,q) = \frac{1}{1-\dfrac{\lambda^2(1-q)\psi(\lambda,q^2)}{(1+\lambda^2)\psi(\lambda,q)}}. \tag{3.6.2}$$

Using Heine's transformation formula given in (3.4.3), we transform the basic hypergeometric functions given in (3.6.2) as follows

$$\begin{aligned}
\psi(\lambda,q) &= {}_2\phi_1(i\lambda\sqrt{q}, -i\lambda\sqrt{q}; \lambda^2 q; q, q) && (3.6.3)\\
&= \frac{(-i\lambda\sqrt{q};q)_\infty (i\lambda q^{3/2};q)_\infty}{(\lambda^2 q;q)_\infty (q;q)_\infty}\, {}_2\phi_1(i\lambda\sqrt{q}, q; i\lambda q^{3/2}; q, -i\lambda\sqrt{q}) && (3.6.4)
\end{aligned}$$

and

$$\psi(\lambda,q^2) = {}_2\phi_1(i\lambda\sqrt{q},-i\lambda\sqrt{q};\lambda^2 q;q,q^2) \tag{3.6.5}$$

$$= \frac{(-i\lambda\sqrt{q};q)_\infty(i\lambda q^{5/2};q)_\infty}{(\lambda^2 q;q)_\infty(q^2;q)_\infty}\,{}_2\phi_1(i\lambda\sqrt{q},q^2;i\lambda q^{5/2};q,-i\lambda\sqrt{q}). \tag{3.6.6}$$

Substituting the transformations (3.6.4) and (3.6.6) in the half plane limit (3.6.2) yields

$$G'(t,q) = \cfrac{1}{1-\cfrac{\lambda^2(1-q)\frac{(-i\lambda\sqrt{q};q)_\infty(i\lambda q^{5/2};q)_\infty}{(\lambda^2 q;q)_\infty(q^2;q)_\infty}\,{}_2\phi_1(i\lambda\sqrt{q},q^2;i\lambda q^{5/2};q,-i\lambda\sqrt{q})}{(1+\lambda^2)\frac{(-i\lambda\sqrt{q};q)_\infty(i\lambda q^{3/2};q)_\infty}{(\lambda^2 q;q)_\infty(q;q)_\infty}\,{}_2\phi_1(i\lambda\sqrt{q},q;i\lambda q^{3/2};q,-i\lambda\sqrt{q})}}. \tag{3.6.7}$$

Simplifying the expression we obtain

$$G'(t,q) = \cfrac{1}{1-\cfrac{\lambda^2(1-q)^2\sum_{k=0}^{\infty}\frac{(i\lambda\sqrt{q};q)_k(q^2;q)_k}{(i\lambda q^{5/2};q)_k(q;q)_k}(-i\lambda\sqrt{q})^k}{(1+\lambda^2)(1-i\lambda q^{3/2})\sum_{k=0}^{\infty}\frac{(i\lambda\sqrt{q};q)_k(q;q)_k}{(i\lambda q^{3/2};q)_k(q;q)_k}(-i\lambda\sqrt{q})^k}} \tag{3.6.8}$$

$$= \cfrac{1}{1-\cfrac{\lambda^2(1-q)\sum_{k=0}^{\infty}\frac{(1-q^{k+1})}{(1-i\lambda q^{k+1/2})(1-i\lambda q^{k+3/2})}(-i\lambda\sqrt{q})^k}{(1+\lambda^2)\sum_{k=0}^{\infty}\frac{(-i\lambda\sqrt{q})^k}{(1-i\lambda q^{k+1/2})}}}. \tag{3.6.9}$$

To simplify further we consider the expression in (3.6.9). We aim to simplify the terms in the sums on the right hand side of (3.6.9). For this we let

$$N = \sum_{k=0}^{\infty}\frac{(1-q^{k+1})}{(1-i\lambda q^{k+1/2})(1-i\lambda q^{k+3/2})}(-i\lambda\sqrt{q})^k \tag{3.6.10}$$

and

$$D = \frac{(-i\lambda\sqrt{q})^k}{(1 - i\lambda q^{k+1/2})} \ . \tag{3.6.11}$$

We substitute $x = -i\lambda\sqrt{q}$ in N and apply partial fraction decomposition to get

$$N = \sum_{k=0}^{\infty} \left(\frac{x^k q(x+1)}{x(1+xq^k q)(q-1)} - \frac{x^k(q+x)}{x(1+xq^k q)(q-1)} \right) \ . \tag{3.6.12}$$

Shifting summation indices and combining the fractions, we find

$$N = \frac{(q-x^2)\sum_{k=0}^{\infty} \frac{x^k}{1+xq^k}}{x^2(q-1)} - \frac{q}{x^2(q-1)} \ . \tag{3.6.13}$$

Similarly we consider the expression D and substitute $x = -i\lambda\sqrt{q}$ to get

$$D = \sum_{k=0}^{\infty} \frac{x^k}{1+xq^k} \ . \tag{3.6.14}$$

Substituting (3.6.13) and (3.6.14) in (3.6.9) and simplifying, we get the final result as

$$G'(t,q) = (1+\lambda^2) \sum_{k=0}^{\infty} \frac{(-i\lambda\sqrt{q})^k}{(1 - i\lambda\sqrt{q}q^k)}. \tag{3.6.15}$$

This gives the desired result. □

3.6.1 An explicit formula for q-secant numbers

Next we extract the coefficient of t^{2N} of $G'(t,q)$ given in (3.6.1). Our result is in agreement with Theorem 1.5 in [11] where relabelling $k = m$, $n = N$, $i = l + m$ and simplifying results in the expression (3.6.16).

Corollary 3.9. $Q'_N(q) = [t^{2N}]G'(t,q)$ is given by

$$Q'_N(q) = \frac{1}{(1-q)^{2N}} \sum_{m=0}^{N} \frac{q^{m^2+m}\left(\sum_{l=-m}^{m}(-1)^l q^{-l^2}\right)(2m+1)\binom{2N}{N+m}}{N+m+1}. \tag{3.6.16}$$

Proof. To prove this corollary we will again use the Lemma 3.6. The sum in (3.6.1) can be identified with $R(-i\lambda\sqrt{q}, i\lambda\sqrt{q}, q)$, so we get

$$G'(t,q) = (1+\lambda)^2 \sum_{k=0}^{\infty} \frac{q^{n^2+n}\lambda^{2n}(1-\lambda^2 q^{2n+1})}{(1+\lambda^2 q^{2n+1})}. \tag{3.6.17}$$

We remind that t-dependence on the right hand side is implicit in $\lambda = \lambda(t)$. To extract the coefficient of t^{2N}, we evaluate the contour integral

$$[t^{2N}]G'(t,q) = \frac{1}{2\pi i}\oint \frac{(1+\lambda(t))^2 \sum_{k=0}^{\infty} \frac{q^{n^2+n}\lambda(t)^{2n}(1-\lambda(t)^2 q^{2n+1})}{(1+\lambda(t)^2 q^{2n+1})}}{t^{2N+1}} dt. \tag{3.6.18}$$

From (3.4.26) substituting the value of $dt/d\lambda$ we get

$$[t^{2N}]G'(t,q) = \frac{1}{2\pi i}\oint \left(\frac{(1+\lambda)^{2N}\left(\sum_{k=0}^{\infty} \frac{q^{n^2+n}\lambda^{2n}(1-\lambda^2 q^{2n+1})}{(1+\lambda^2 q^{2n+1})}\right)(1-\lambda^2)}{\lambda^{2N}(1-q)^{2N})}\right)\frac{d\lambda}{\lambda}. \tag{3.6.19}$$

The expression above is in terms of λ. Using the relationship between λ and t we know

$$[t^{2N}]G'(t,q) = [\lambda^0]H'_N(\lambda,q) \tag{3.6.20}$$

where

$$H'_N(\lambda, q) = \frac{\left(\lambda + \frac{1}{\lambda}\right)^{2N} (1 - \lambda^2) \left(\sum_{k=0}^{\infty} \frac{q^{n^2+n}\lambda^{2n}(1 - \lambda^2 q^{2n+1})}{(1 + \lambda^2 q^{2n+1})} \right)}{(1 - q)^{2N}}. \tag{3.6.21}$$

From the expression $H'_N(\lambda, q)$ we can extract the constant term in λ as follows

$$[\lambda^0] H'_N(\lambda, q) = [\lambda^0] T_1 \sum_{k=0}^{\infty} T_2 \ , \tag{3.6.22}$$

where

$$T_1 = \frac{\left(\lambda + \frac{1}{\lambda}\right)^{2N} (1 - \lambda^2)}{(1 - q)^{2N}}$$

and

$$T_2 = \frac{q^{n^2+n}\lambda^{2n}(1 - \lambda^2 q^{2n+1})}{(1 + \lambda^2 q^{2n+1})}. \tag{3.6.23}$$

We can express T_1 and T_2 as

$$T_1 = \frac{\sum_{k=0}^{2N+1} \frac{(2N)!(2N - 2k + 1)\lambda^{2k-2N}}{k!(2N - k + 1)!}}{(1 - q)^{2N}} \tag{3.6.24}$$

and

$$T_2 = q^{n^2+n}\lambda^{2n} \left(2 \sum_{l=0}^{\infty} (-1)^l (\lambda^2 q^{2n+1})^l - 1 \right). \tag{3.6.25}$$

Substituting the expression (3.6.24) and (3.6.25) in (3.6.22) gives

$$H'_N(\lambda, q) = \frac{1}{(1 - q)^{2N}} \left(\sum_{k=0}^{2N+1} \frac{(2N)!(2N - 2k + 1)\lambda^{2k-2N}}{k!(2N - k + 1)!} \sum_{n=0}^{\infty} q^{n^2+n}\lambda^{2n} \left(2 \sum_{l=0}^{\infty} (-1)^l (\lambda^2 q^{2n+1})^l - 1 \right) \right). \tag{3.6.26}$$

Expanding the terms and combining the summation implies

$$H'_N(\lambda, q) = \frac{1}{(1-q)^{2N}}$$
$$\left(2\sum_{k=0}^{2N+1}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l q^{n^2+n+2nl+l}\frac{(2N)!(2N-2k+1)\lambda^{2k-2N+2n+2l}}{k!(2N-k+1)!}\right.$$
$$\left. - \sum_{k=0}^{2N+1}\sum_{n=0}^{\infty} q^{n^2+n}\frac{(2N)!(2N-2k+1)\lambda^{2k-2N+2n}}{k!(2N-k+1)!}\right). \quad (3.6.27)$$

We aim to get the coefficient of λ^0, so we combine the powers of λ and equate them to 0 to find k. For the first term we equate $2k-2N+2n+2l=0$ and get $k=N-n-l$. For the second term we let $2k-2N+2n=0$ and get $k=N-n$. Inserting the value of k in (3.6.27) we get

$$Q'_N(q) = \frac{1}{(1-q)^{2N}}\left(2\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l q^{n^2+n+2nl+l}\frac{(2N)!(2n+2l+1)}{(N-n-l)!(N+n+l+1)!}\right.$$
$$\left. - \sum_{n=0}^{\infty} q^{n^2+n}\frac{(2N)!(2n+1)}{(N-n)!(N+n+1)!}\right). \quad (3.6.28)$$

We add and subtract the term $-l^2$ to the power of q in the first term to complete the square. We obtain

$$Q'_N(q) = \frac{1}{(1-q)^{2N}}\left(2\sum_{n=0}^{N}\sum_{l=0}^{N-n}(-1)^l q^{(n+l)^2+n+l} q^{-l^2}\frac{(2N)!(2n+2l+1)}{(N-n-l)!(N+n+l+1)!}\right.$$
$$\left. - \sum_{n=0}^{N} q^{n^2+n}\frac{(2N)!(2n+1)}{(N-n)!(N+n+1)!}\right). \quad (3.6.29)$$

The sum runs over $n+l \leq N$, so we substitute $n+l=m$ which simplifies the expression as

$$Q'_N(q) = \frac{1}{(1-q)^{2N}} \left(2\sum_{m=0}^{N}\sum_{l=0}^{m}(-1)^l q^{m^2+m} q^{-l^2} \frac{(2N)!(2m+1)}{(N-m)!(N+m+1)!} \right.$$
$$\left. - \sum_{n=0}^{N} q^{n^2+n} \frac{(2N)!(2n+1)}{(N-n)!(N+n+1)!} \right). \quad (3.6.30)$$

Changing the summation of variable in the second term from n to m and simplifying gives

$$Q'_N(q) = \frac{1}{(1-q)^{2N}} \left(\sum_{m=0}^{N} q^{m^2+m} \frac{(2N)!(2m+1)}{(N-m)!(N+m+1)!} \left(2\sum_{l=0}^{m}(-1)^l q^{-l^2} - 1 \right) \right). \quad (3.6.31)$$

The final result is given by (3.6.16). □

3.7 Identities

The central results of this chapter have been given in Theorems 3.3 and 3.7, which express finite continued fractions in terms of basic hypergeometric functions. For example, for q-tangent numbers we have

$$\cfrac{1}{1-\cfrac{\alpha^2(1-q)(1-q^2)}{1-\cfrac{\alpha^2(1-q^2)(1-q^3)}{\ddots-\cfrac{\alpha^2(1-q^{w-1})(1-q^w)}{1-\alpha^2(1-q^w)(1-q^{w+1})}}}} =$$

$$\cfrac{1}{1-\cfrac{\lambda^2(1-q)\left[\bar{\lambda}^w\phi(\lambda,q^3)\phi\left(\bar{\lambda},q^{w+3}\right)-\lambda^w\phi\left(\bar{\lambda},q^3\right)\phi(\lambda,q^{w+3})\right]}{(1+\lambda^2)\left[\bar{\lambda}^w\phi(\lambda,q^2)\phi\left(\bar{\lambda},q^{w+3}\right)-\lambda^{w+2}\phi\left(\bar{\lambda},q^2\right)\phi(\lambda,q^{w+3})\right]}}$$

and a similar result holds for q-secant numbers. The point we would like to make in this section is that these results can be re-interpreted as giving hierarchies of identities for basic hypergeometric functions. For w small, the left hand side is a relatively simple rational function in t and q, whereas the right hand side is a weighted ratio of products of basic hypergeometric functions at specific arguments. We make the resulting identities explicit for $w = 1$ in the following corollary.

Corollary 3.10.

$$\frac{1-q^2}{1-\nu^2} = \frac{\left[\begin{array}{l} {}_2\phi_1(\nu,-\nu;-\nu^2q;q,q^3)\,{}_2\phi_1\left(-\bar{\nu},\bar{\nu};-\nu^2\bar{q};q,q^4\right) \\ \quad +\nu\,{}_2\phi_1\left(-\bar{\nu},\bar{\nu};-\nu^2\bar{q};q,q^3\right)\,{}_2\phi_1(\nu,-\nu;-\nu^2q;q,q^4)\end{array}\right]}{\left[\begin{array}{l} {}_2\phi_1(\nu,-\nu;-\nu^2q;q,q^2)\,{}_2\phi_1\left(-\bar{\nu},\bar{\nu};-\nu^2\bar{q};q,q^4\right) \\ \quad -\nu^4\,{}_2\phi_1\left(-\bar{\nu},\bar{\nu};-\nu^2\bar{q};q,q^2\right)\,{}_2\phi_1(\nu,-\nu;-\nu^2q;q,q^4)\end{array}\right]} \tag{3.7.1}$$

where $\nu = i\lambda$, $\bar{\nu} = \frac{1}{\nu}$ and $\bar{q} = \frac{1}{q}$, and

$$\frac{1-q}{1-\mu^2\bar{q}} = \frac{\left[\begin{array}{l} {}_2\phi_1(\mu,-\mu;-\mu^2;q,q^2)\,{}_2\phi_1\left(-q\bar{\mu},q\bar{\mu};-q^2\bar{\mu}^2;q,q^3\right) \\ \quad +(\mu^2\bar{q})\,{}_2\phi_1\left(-q\bar{\mu},q\bar{\mu};-q^2\bar{\mu}^2;q,q^2\right)\,{}_2\phi_1(\mu,-\mu;-\mu^2;q,q^3)\end{array}\right]}{\left[\begin{array}{l} {}_2\phi_1(\mu,-\mu;-\mu^2;q,q)\,{}_2\phi_1\left(-q\bar{\mu},q\bar{\mu};-q^2\bar{\mu}^2;q,q^3\right) \\ \quad -\mu^4\bar{q}^2\,{}_2\phi_1\left(-q\bar{\mu},q\bar{\mu};-q^2\bar{\mu}^2;q,q\right)\,{}_2\phi_1(\mu,-\mu;-\mu^2;q,q^3)\end{array}\right]} \tag{3.7.2}$$

where $\mu = i\lambda\sqrt{q}$, $\bar{\mu} = \frac{1}{\mu}$ and $\bar{q} = \frac{1}{q}$.

Proof. Insert $w = 1$ in Theorems 3.3 and 3.7 and simplify. $\square$

To the best of our knowledge these identities are new. It would be interesting to find an alternative derivation and perhaps deeper understanding of their meaning.

Chapter 4

Generalised Weighted Paths

4.1 Introduction and model definition

Consider a directed path in a slit $\mathbb{Z} \times \{0, 1, \cdots, w\}$ of width w, starting at point $(0, u)$ and ending at point (n, v), taking its steps from $\{1\} \times S$, where $S \subset \mathbb{Z}$ is a finite set. For simplicity we call S the step set. We define $A = S \cap \mathbb{Z}_0^+$ and $B = -(S \backslash A)$. Figure 4.1 shows such a generalised path.

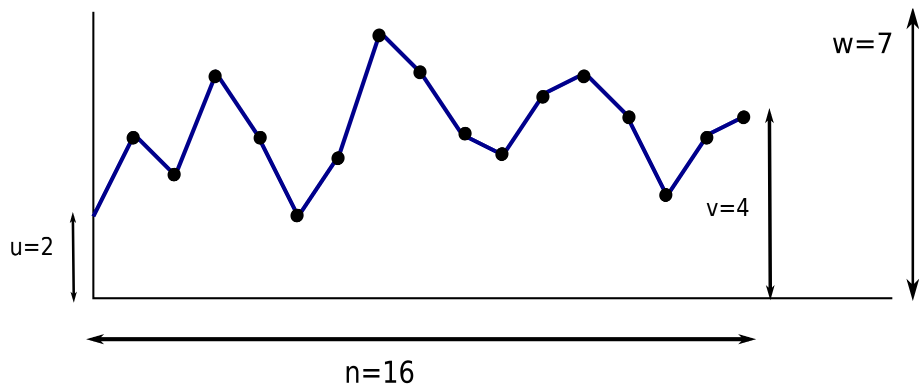

Figure 4.1: Generalised path of length $n = 16$ with northeast steps $A = \{1, 3, 4, 5, 6\}$ and southeast steps $B = \{1, 2, 3, 4\}$ in a slit of width 7, starting at height 2 and ending at height 4.

We are interested in the enumeration of these generalised paths, where we also associate specific weights to individual steps. Every up step in A is weighted by p_a, $a \in A$ and every down step has an associated weight q_b, $b \in B$. If the maximum of A and B are α and β respectively, we assume p_α and q_β to be non zero. The length n of the walk is taken into account by t. In particular, the generating function of generalised weighted paths is given by

$$G(t,z) = \sum_{v=0}^{w} G^{w,\alpha,\beta}_{(u,v)}(t) z^v, \tag{4.1.1}$$

where $G^{w,\alpha,\beta}_{(u,v)}(t)$ is the generating function of the paths and t is conjugate to the length n of the walk, u is the starting height, and v is the ending height of the path.

4.2 The main result

Theorem 4.1. *The generating function $G^{w,\alpha,\beta}_{(u,v)}(t)$ of generalised weighted paths is given by*

$$G^{w,\alpha,\beta}_{(u,v)}(t) = \frac{(-1)^{1-\alpha}}{tp_\alpha} \frac{s_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(\bar{z})}{s_{((w+1)^\alpha,0^\beta)}(\bar{z})} \,, \tag{4.2.1}$$

where $\bar{z}$ are the $\alpha+\beta$ roots of

$$K(t,z) = 1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B} q_b z^{-b} \,,$$

and $s_{\lambda/\mu}(z)$ is a skew Schur function.

We shall prove this theorem in a sequence of steps in the next two sections. We will first write down a functional equation satisfied by the generating function, and define the notion of the kernel of this functional equation, which is essentially a polynomial in z of degree $\alpha+\beta$, related to $K(t,z)$ in the statement of the theorem above. Coefficients of the kernel can be interpreted in terms of elementary symmetric functions of the roots. We cast the enumeration problem in terms of a system of linear equations. Using elementary symmetric functions will allow us to employ the Jacobi-Trudi formula to express the solution of the system in terms of skew Schur functions, leading to the statement of the theorem.

4.3 The functional equation

An n-step walk is constructed by adding steps from the step set S to an $(n-1)$-step walk, provided $n > 0$. We also drop w, α and β from $G_{(u,v)}^{w,\alpha,\beta}$ by writing $G_{(u,v)}^{w,\alpha,\beta}(t) \equiv G_{(u.v)}(t)$. This leads to the functional equation for the generating function $G(t,z)$ given by:

$$\begin{aligned} G(t,z) = z^u + t\left(\sum_{a\in A} p_a z^a + \sum_{b\in B}\frac{q_b}{z^b}\right) G(t,z) \\ - t\sum_{j=1}^{\infty} z^{w+j}\sum_{a\geq j} p_a G_{(u,w-a+j)}(t) - t\sum_{j=1}^{\infty} z^{-j}\sum_{b\geq j} q_b G_{(u,b-j)}(t). \quad (4.3.1)\end{aligned}$$

Here z^u represents the zero step walk starting and ending at height u. The term $t\left(\sum_{a\in A} p_a z^a + \sum_{b\in B}\frac{q_b}{z^b}\right) G(t,z)$ corresponds to steps appended without the consideration of violation of boundaries. The steps not allowed are removed by subtracting the terms which account for the steps crossing the line $y = 0$ and $y = w$. For example $t\sum_{j=1}^{\infty} z^{w+j}\sum_{a\geq j} p_a G_{(u,w-a+j)}(t)$ adjusts the steps going across the line $y = w$ and $t\sum_{j=1}^{\infty} z^{-j}\sum_{b\geq j} q_b G_{(u,b-j)}(t)$ corresponds to steps below the boundary $y = 0$. We rearrange the functional equation as

$$\begin{aligned} \left(1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B}\frac{q_b}{z^b}\right) G(t,z) = \\ z^u - t\sum_{j=1}^{\infty} z^{w+j}\sum_{a\geq j} p_a G_{(u,w-a+j)}(t) - t\sum_{j=1}^{\infty} z^{-j}\sum_{b\geq j} q_b G_{(u,b-j)}(t). \quad (4.3.2)\end{aligned}$$

This gives us the kernel $K(t,z)$ of the functional equation

$$K(t,z) = 1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B} q_b z^{-b}. \tag{4.3.3}$$

We finish this section proving a lemma relating the coefficients of the kernel to elementary symmetric functions.

Lemma 4.2. The kernel can be written as

$$K(t,z) = -tp_\alpha \sum_{i=0}^{\alpha+\beta} z^{\alpha-i}(-1)^i e_i(z_1, z_2, \ldots, z_{\alpha+\beta}) \tag{4.3.4}$$

where $\bar{z} = z_1, z_2, \ldots, z_{\alpha+\beta}$ are the roots of the kernel $K(t,z)$, and we have

$$-tp_a = -tp_\alpha(-1)^{\alpha-a} e_{\alpha-a}(\bar{z}) \tag{4.3.5}$$

$$1 - tp_0 = -tp_\alpha(-1)^{\alpha} e_{\alpha}(\bar{z}) \tag{4.3.6}$$

$$-tq_b = -tp_\alpha(-1)^{\alpha+b} e_{\alpha+b}(\bar{z}) \tag{4.3.7}$$

for $1 \le a \le \alpha$ and $1 \le b \le \beta$.

Proof. We simplify the kernel and write as follows

$$K(t,z) = \frac{-tp_\alpha}{z^\beta}\left(-\frac{z^\beta}{tp_\alpha} + \sum_{a=0}^{\alpha} \frac{p_a}{p_\alpha} z^{a+\beta} + \sum_{b=1}^{\beta} \frac{q_b}{p_\alpha} z^{\beta-b}\right) = \frac{-tp_\alpha}{z^\beta}\prod_{k=1}^{\alpha+\beta}(z - z_k). \tag{4.3.8}$$

We can relate the coefficients of the polynomial given by the kernel to the sum and product of roots. Using the following relation we can transform the kernel.

$$\prod_{k=1}^{n}(z + z_k) = \sum_{\alpha=0}^{n} z^{n-\alpha} e_\alpha(\bar{z}) = z^n e_0 + z^{n-1} e_1 + \cdots + e_n\,. \tag{4.3.9}$$

The kernel expressed in terms of elementary symmetric functions is given by

$$K(t,z) = 1-t\sum_{a=0}^{\alpha} p_a z^a - t\sum_{b=1}^{\beta} q_b z^{-b} = \frac{-tp_\alpha}{z^\beta}\sum_{i=0}^{\alpha+\beta} z^{\alpha+\beta-i}(-1)^i e_i(\bar{z}) = -tp_\alpha \sum_{i=0}^{\alpha+\beta} z^{\alpha-i}(-1)^i e_i(\bar{z}). \tag{4.3.10}$$

This implies

$$1 - t\sum_{a=0}^{\alpha} p_a z^a - t\sum_{b=1}^{\beta} q_b z^{-b} = -tp_\alpha \sum_{i=0}^{\alpha+\beta} z^{\alpha-i}(-1)^i e_i(\bar{z}) \ . \tag{4.3.11}$$

Next we compare coefficients in (4.3.11) for different powers of z. For $\alpha \geq a \geq 1$, let $a = \alpha - i$ in right hand side of (4.3.11) to get

$$-\, tp_a = -tp_\alpha (-1)^{\alpha-a} e_{\alpha-a}(\bar{z}) \ . \tag{4.3.12}$$

For $a = 0$, let $i = \alpha$,

$$1 - tp_0 = -tp_\alpha (-1)^{\alpha} e_{\alpha}(\bar{z}) \ . \tag{4.3.13}$$

For $\beta \geq b \geq 1$, we substitute $-b = \alpha - i$

$$-\, tq_b = -tp_\alpha (-1)^{\alpha+b} e_{\alpha+b}(\bar{z}) \ . \tag{4.3.14}$$

□

4.4 Solution of the functional equation

We aim to rewrite the functional equation (4.3.2) in terms of elementary symmetric functions instead of weights p_a, q_b and t. For that we substitute the expression for the kernel from Lemma 4.2 into it. This gives

$$\left(-tp_\alpha \sum_{i=0}^{\alpha+\beta} z^{\alpha-i}(-1)^i e_i\right) \sum_{v=0}^{w} G_{(u,v)}(t) z^v = \\ z^u - t\sum_{j=1}^{\infty} z^{w+j} \sum_{a\geq j} p_a G_{(u,w-a+j)}(t) \\ - t\sum_{j=1}^{\infty} z^{-j} \sum_{b\geq j} q_b G_{(u,b-j)}(t)\ . \quad (4.4.1)$$

Similarly we express tp_a and tq_b in terms of elementary symmetric functions given by (4.3.5) and (4.3.7) in Lemma 4.2. Rearranging terms and multiplying throughout by $-1/tp_\alpha$ we get

$$\sum_{v=0}^{w}\sum_{i=0}^{\alpha+\beta} z^{v+\alpha-i}(-1)^i e_i G_{(u,v)}(t) = \\ -\frac{z^u}{tp_\alpha} + \sum_{j=1}^{\infty}\sum_{a\geq j} z^{w+j}(-1)^{\alpha-a} e_{\alpha-a} G_{(u,w-a+j)}(t) \\ + \sum_{j=1}^{\infty}\sum_{b\geq j} z^{-j}(-1)^{\alpha+b} e_{\alpha+b} G_{(u,b-j)}(t). \quad (4.4.2)$$

Next we rearrange terms on the left hand side to better extract the coefficient of z^v,

$$\sum_{i=0}^{\alpha+\beta}\sum_{v=\alpha-i}^{w+\alpha-i} (-1)^i e_i G_{(u,v-\alpha+i)}(t) z^v = \\ -\frac{z^u}{tp_\alpha} + \sum_{j=1}^{\infty}\sum_{a\geq j} z^{w+j}(-1)^{\alpha-a} e_{\alpha-a} G_{(u,w-a+j)}(t) \\ + \sum_{j=1}^{\infty}\sum_{b\geq j} z^{-j}(-1)^{\alpha+b} e_{\alpha+b} G_{(u,b-j)}(t). \quad (4.4.3)$$

As G_v is zero for $v < 0$ and $v > w$, so we extend the limits of summation of v and write the functional equation as

$$\sum_{i=0}^{\alpha+\beta} \sum_{v=-\infty}^{\infty} (-1)^i e_i G_{(u,v-\alpha+i)} z^v = \\ -\frac{z^u}{tp_\alpha} + \sum_{j=1}^{\infty} \sum_{a=j}^{\alpha} z^{w+j} (-1)^{\alpha-a} e_{\alpha-a} G_{(u,w-a+j)}(t) + \\ \sum_{j=1}^{\infty} \sum_{b=j}^{\infty} z^{-j} (-1)^{\alpha+b} e_{\alpha+b} G_{(u,b-j)}(t). \quad (4.4.4)$$

Changing the order of summation and restricting the sum to non zero terms where appropriate, we have

$$\sum_{v=-\infty}^{\infty} \left(\sum_{i=0}^{\alpha+\beta} (-1)^i e_i G_{(u,v-\alpha+i)}(t) \right) z^v = \\ -\frac{z^u}{tp_\alpha} + \sum_{a=1}^{\alpha} \sum_{j=1}^{a} z^{w+j} (-1)^{\alpha-a} e_{\alpha-a} G_{(u,w-a+j)}(t) \\ + \sum_{b=1}^{\beta} \sum_{j=1}^{b} z^{-j} (-1)^{\alpha+b} e_{\alpha+b} G_{(u,b-j)}(t). \quad (4.4.5)$$

On the right hand side, we let $w + j = v$ in the first sum and $v = -j$ in the second sum, and write the summation for v as follows

$$\sum_{v=-\infty}^{\infty} \left(\sum_{i=0}^{\alpha+\beta} (-1)^i e_i G_{(u,v-\alpha+i)}(t) \right) z^v = \\ -\frac{z^u}{tp_\alpha} + \sum_{a=1}^{\alpha} \sum_{v=w+1}^{w+a} z^v (-1)^{\alpha-a} e_{\alpha-a} G_{(u,v-a)}(t) \\ + \sum_{b=1}^{\beta} \sum_{v=-b}^{-1} z^v (-1)^{\alpha+b} e_{\alpha+b} G_{(u,v+b)}(t). \quad (4.4.6)$$

Exchanging the order of summation, we now can write the outer summation uniformly as sums over z^v as follows

$$\sum_{v=-\infty}^{\infty}\left(\sum_{i=0}^{\alpha+\beta}(-1)^i e_i G_{(u,v-\alpha+i)}(t)\right) z^v = \\ -\frac{z^u}{tp_\alpha} + \sum_{v=w+1}^{w+\alpha}\left(\sum_{a=v-w}^{\alpha}(-1)^{\alpha-a} e_{\alpha-a} G_{(u,v-a)}(t)\right) z^v \\ + \sum_{v=-\beta}^{-1}\left(\sum_{b=v}^{\beta} z^v(-1)^{\alpha+b} e_{\alpha+b} G_{(u,v+b)}(t)\right) z^v. \quad (4.4.7)$$

Let $\alpha - a = i$ for the first summation and $\alpha + b = i$ for second summation. We get

$$\sum_{v=-\infty}^{\infty}\left(\sum_{i=0}^{\alpha+\beta}(-1)^i e_i G_{(u,v-\alpha+i)}(t)\right) z^v = \\ -\frac{z^u}{tp_\alpha} + \sum_{v=w+1}^{w+\alpha}\left(\sum_{i=0}^{\alpha+(w-v)}(-1)^i e_i G_{(u,v-\alpha+i)}(t)\right) z^v \\ + \sum_{v=-\beta}^{-1}\left(\sum_{i=\alpha+v}^{\alpha_+\beta}(-1)^i e_i G_{(u,v-\alpha+i)}(t)\right) z^v. \quad (4.4.8)$$

Upon inspection one can now see that the sums on the right hand side can be matched by sums over identical terms on the left hand side. The boundary corrections in the functional equation have of course been introduced to precisely that effect, as they were added to correct for steps that went beyond the upper and lower boundaries. We are left with with the following expression

$$\sum_{v=0}^{w}\left(\sum_{i=0}^{\alpha+\beta}(-1)^i e_i G_{(u,v-\alpha+i)}(t)\right) z^v = -\frac{z^u}{tp_\alpha} \quad (4.4.9)$$

Comparing coefficients of z^v for $0 \leq v \leq w$, Equation (4.4.9) is equivalent to a system of $w+1$ equations. This can be written in matrix form as

$$\begin{bmatrix} (-1)^{\alpha} e_{\alpha} & (-1)^{\alpha+1} e_{\alpha+1} & \cdots & (-1)^{\alpha+\beta} e_{\alpha+\beta} & \cdots & 0 \\ (-1)^{\alpha-1} e_{\alpha-1} & (-1)^{\alpha} e_{\alpha} & \cdots & (-1)^{\alpha+\beta-1} e_{\alpha+\beta-1} & \cdots & 0 \\ (-1)^{\alpha-2} e_{\alpha-2} & (-1)^{\alpha-1} e_{\alpha-1} & \cdots & (-1)^{\alpha+\beta-2} e_{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_0 & -e_1 & \cdots & (-1)^{\beta} e_{\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{\alpha} e_{\alpha} \end{bmatrix} \begin{bmatrix} G_{(u,0)}(t) \\ G_{(u,1)}(t) \\ G_{(u,2)}(t) \\ \vdots \\ G_{(u,w)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ -\frac{1}{tp_{\alpha}} \\ \vdots \\ 0 \end{bmatrix}, \tag{4.4.10}$$

where $-\frac{1}{tp_{\alpha}}$ is an entry at u^{th} position, with every other entry in the vector on the right hand side being zero. We can evaluate the unknowns $G_{(u,v)}(t)$ for $v = 0 \ldots w$, by using Cramer's rule. We write this matrix equation as

$$\tilde{A}x = b, \tag{4.4.11}$$

where $\tilde{A}$ is the coefficient matrix of dimension $w+1$, x is the matrix of unknowns $G_{(u,v)}(t)$, and b is the column vector on the right hand side with a single non zero entry $-\frac{1}{tp_{\alpha}}$. Note that in $\tilde{A}$ the only zeros come from e_i terms in (4.4.9) with $i < 0$ or $i > \alpha + \beta$, and the non zero entries form a diagonal band. We want to remove the negative signs of the entries in $\tilde{A}$ to write the matrix equation in terms of the matrix

$$A = \begin{bmatrix} e_{\alpha} & e_{\alpha+1} & e_{\alpha+2} & \cdots & e_{\alpha+\beta} & \cdots & 0 \\ e_{\alpha-1} & e_{\alpha} & e_{\alpha+1} & \cdots & e_{\alpha+\beta-1} & \cdots & 0 \\ e_{\alpha-2} & e_{\alpha-1} & e_{\alpha} & \cdots & e_{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_0 & e_1 & e_2 & \cdots & e_{\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & e_{\alpha} \end{bmatrix}. \tag{4.4.12}$$

We accomplish this by applying a transformation given by the matrix

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{w+1} \end{bmatrix}. \tag{4.4.13}$$

For the matrix S we know that $S = S^{-1}$. The matrix equation (4.4.11) will be transformed as $S\tilde{A}S^{-1}Sx = Sb$. We note that $S\tilde{A}S^{-1} = (-1)^{\alpha}A$ and $Sb = (-1)^{u}b$, so we have

$$(-1)^{\alpha}A(Sx) = (-1)^{u}b, \tag{4.4.14}$$

where $(Sx)_k = (-1)^k G_{(u,k)}(t)$. To evaluate Sx let $A_{(u,v)}$ be the matrix formed by replacing column v in A with the column vector $(-1)^{\alpha}Sb$, which has $(-1)^{u+1-\alpha}\frac{1}{tp_{\alpha}}$ at position u. As per Cramer's rule we can say that

$$(-1)^{v}G_{(u,v)}(t) = \frac{|A_{(u,v)}|}{|A|}. \tag{4.4.15}$$

Using the second Jacobi -Trudi formula, we can write the determinant of the matrix A in terms of Schur functions. The second Jacobi-Trudi formula expresses the Schur function as a determinant in terms of the elementary symmetric functions,

$$s_{\lambda} = \det(e_{\lambda'_i+j-i})_{i,j=1}^{l(\lambda')} = \begin{vmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & e_{\lambda'_1+2} & \cdots & e_{\lambda'_1+l-1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & e_{\lambda'_2+1} & \cdots & e_{\lambda'_1+l-2} \\ e_{\lambda'_3-2} & e_{\lambda'_3-1} & e_{\lambda'_3} & \cdots & e_{\lambda'_1+l-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_l-l+1} & e_{\lambda'_l-l+2} & e_{\lambda'_l-l+3} & \cdots & e_{\lambda'_l} \end{vmatrix}. \tag{4.4.16}$$

Comparing the determinant in (4.4.16) with the matrix A in (4.4.12), we can see that the conjugate partition λ' is given by

$$\lambda' = \left(\alpha^{w+1}\right). \tag{4.4.17}$$

From this conjugate partition we can write $\lambda = ((w+1)^{\alpha}, 0^{\beta})$ and so the determinant of the matrix A can be written as

$$|A| = s_{((w+1)^{\alpha},0^{\beta})}(z_1, z_2, \cdots, z_{\alpha+\beta}). \tag{4.4.18}$$

Note that we have chosen the convention to let the partition have the same number of parts as we have roots $z_1, z_2, \cdots, z_{\alpha+\beta}$, so that we supplement the partition with zero size parts as needed.

Further we need to evaluate $G_{(u,v)}(t)$ for $v = 0 \dots w$. Recall that the matrix $A_{(u,v)}$ is equal to

$$\begin{bmatrix} e_{\alpha} & e_{\alpha+1} & \cdots & e_{\alpha+v-1} & 0 & \cdots & e_{\alpha+\beta} & \cdots & 0 \\ e_{\alpha-1} & e_{\alpha} & \cdots & e_{\alpha+v-2} & 0 & \cdots & e_{\alpha+\beta-1} & \cdots & 0 \\ e_{\alpha-2} & e_{\alpha-1} & \cdots & e_{\alpha+v-3} & 0 & \cdots & e_{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_{\alpha-u+1} & e_{\alpha-u+2} & \cdots & e_{\alpha+v-u} & 0 & \cdots & e_{\alpha+\beta-u+1} & \cdots & 0 \\ e_{\alpha-u} & e_{\alpha-u+1} & \cdots & e_{\alpha+v-u-1} & (-1)^{u+1-\alpha}\frac{1}{tp_{\alpha}} & \cdots & e_{\alpha+\beta-u} & \cdots & 0 \\ e_{\alpha-u-1} & e_{\alpha-u} & \cdots & e_{\alpha+v-u-2} & 0 & \cdots & e_{\alpha+\beta-u-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_0 & e_1 & \cdots & e_{v-1} & 0 & \cdots & e_{\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & e_{\alpha} \end{bmatrix}_{[w+1]}. \tag{4.4.19}$$

To evaluate the determinant of $A_{(u,v)}$ we expand the matrix (4.4.19) by the v^{th}

column. This gives a matrix of size w as follows

$$|A_{(u,v)}| = \frac{(-1)^{2u+v+1-\alpha}}{tp_\alpha}$$
$$\times \begin{vmatrix} e_\alpha & e_{\alpha+1} & \cdots & e_{\alpha+v-1} & e_{\alpha+v+1} & \cdots & e_{\alpha+\beta} & \cdots & 0 \\ e_{\alpha-1} & e_\alpha & \cdots & e_{\alpha+v-2} & e_{\alpha+v} & \cdots & e_{\alpha+\beta-1} & \cdots & 0 \\ e_{\alpha-2} & e_{\alpha-1} & \cdots & e_{\alpha+v-3} & e_{\alpha+v-1} & \cdots & e_{\alpha+\beta-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_{\alpha-u+1} & e_{\alpha-u+2} & \cdots & e_{\alpha+v-u} & e_{\alpha+v-u+2} & \cdots & e_{\alpha+\beta-u+1} & \cdots & 0 \\ e_{\alpha-u-1} & e_{\alpha-u} & \cdots & e_{\alpha+v-u-2} & e_{\alpha+v-u-1} & \cdots & e_{\alpha+\beta-u-1} & \cdots & 0 \\ e_{\alpha-u-2} & e_{\alpha-u-1} & \cdots & e_{\alpha+v-u-3} & e_{\alpha+v-u-1} & \cdots & e_{\alpha+\beta-u-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_0 & e_1 & \cdots & e_{v-1} & e_{v+1} & \cdots & e_\beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & e_\alpha \end{vmatrix}_{[w]} . \tag{4.4.20}$$

To study this matrix we look at the definition of skew Schur functions in terms of the second Jacobi-Trudi formula.

$$s_{\lambda/\mu} = \det(e_{\lambda'_i-\mu'_j+j-i})_{i,j=1}^{l(\lambda')}$$
$$= \begin{vmatrix} e_{\lambda'_1-\mu'_1} & e_{\lambda'_1-\mu'_2+1} & e_{\lambda'_1-\mu'_3+2} & \cdots & e_{\lambda'_1-\mu'_l+l-1} \\ e_{\lambda'_2-\mu'_1-1} & e_{\lambda'_2-\mu'_2} & e_{\lambda'_2-\mu'_3+1} & \cdots & e_{\lambda'_1-\mu'_l+l-2} \\ e_{\lambda'_3-\mu'_1-2} & e_{\lambda'_3-\mu'_2-1} & e_{\lambda'_3-\mu'_3} & \cdots & e_{\lambda'_1-\mu'_l+l-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_l-\mu'_1-l+1} & e_{\lambda'_l-\mu'_2-l+2} & e_{\lambda'_l-\mu'_3-l+3} & \cdots & e_{\lambda'_l-\mu'_l} \end{vmatrix} . \tag{4.4.21}$$

Comparing the determinant of $A_{(u,v)}$ to this determinant we can write the conjugate partitions for both λ and μ accordingly. To determine the conjugate partition of λ, we note that the entries along a column initially decrease by one, so that λ'_i remains unchanged until the u-th row, where a decrease by two implies that λ'_i decreases by one. Hereafter, λ'_i remains constant. Therefore $\lambda' = (c^u, (c-1)^{w-u})$ for some $c > 0$. To determine μ, we consider the change along a row: the entries increase by one except for a jump of two in the v-th column. Therefore $\mu' = (d^v, (d-1)^{w-v})$ for some $d > 0$. From the entry in the first row and column we see that $\lambda'_1 - \mu'_1 = \alpha$, hence $c = d + \alpha$. This determines λ and μ up to a constant. Letting $d = 1$, we find $\lambda' = (\alpha + 1^u, \alpha^{w-u})$ and $\mu' = (1^v, 0^{w-v})$. The partitions λ and μ are thus given by

$$\lambda = (w^\alpha, u, 0^{\beta-1}) \tag{4.4.22}$$

and

$$\mu = (v, 0^{\alpha+\beta-1}), \tag{4.4.23}$$

where we have again added zero size parts to follow the convention established above. The corresponding skew Schur function is

$$s_{(w^\alpha, u, 0^{\beta-1})/(v, 0^{\alpha+\beta-1})}(\bar{z}) \ . \tag{4.4.24}$$

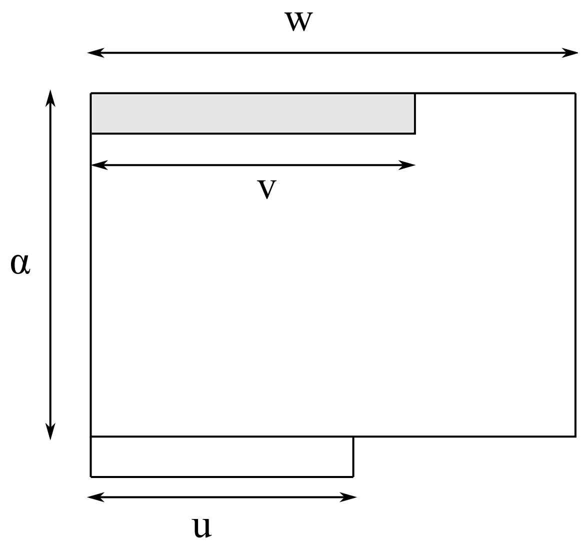

Figure 4.2: Skew partition $\lambda/\mu = (w^\alpha, u, 0^{\beta-1})/(v, 0^{\alpha+\beta-1})$ for the skew Schur function related to $\det A_{(u,v)}$. Here and in what follows we employ the 'British' convention that the parts of the partition are depicted such that the largest part is at the top and the smallest one at the bottom. Note that we do not show parts of zero size.

A pictorial representation of the skew partition is given in Figure 4.2. We see that the associated skew partition is given by a rectangle of size $w \times \alpha$ which has a row of size u added below and a row of size v removed from the top row.

Coming back to Cramer's rule, we see that the determinant of any $A_{(u,v)}$ is given by

$$|A_{(u,v)}| = (-1)^{v+1-\alpha} \frac{1}{tp_\alpha} s_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(\bar{z}) \ . \tag{4.4.25}$$

Taking the value of $|A|$ from (4.4.18) we can write that $G_{(u,v)}$ is given by

$$(-1)^v G_{(u,v)}(t) = \frac{|A_{(u,v)}|}{|A|} = (-1)^{v+1-\alpha} \frac{1}{tp_\alpha} \frac{s_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(\bar{z})}{s_{((w+1)^\alpha,0^\beta)}(\bar{z})} \ . \tag{4.4.26}$$

The final result is (4.2.1).

4.5 Equivalent result in terms of Schur functions

Schur functions form a linear basis for the space of all symmetric polynomials [25]. We can therefore express the skew Schur function in Theorem 4.1 as a linear combination of Schur fiunctions.

Lemma 4.3. Let $\alpha, \beta, w > 0$. Then for $0 \le u, v \le w$ we have

$$s_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})}(z_1, \ldots, z_{\alpha+\beta}) = \sum_{l=0}^{r} s_{(w^{\alpha-1},w-(v-u)_+-l,(u-v)_++l,0^{\beta-1})}(z_1, \ldots, z_{\alpha+\beta}), \tag{4.5.1}$$

where $r = \min(u, v, w-u, w-v)$.

Proof. From Pieri's rule [25, Corollary 7.15.9], we know that for a skew partition

λ/ν, where ν is a single-part partition (v),

$$s_{\lambda/\nu}(z) = \sum_{\mu} s_\mu(z), \tag{4.5.2}$$

where the sum ranges over all partitions $\mu \subseteq \lambda$ for which λ/μ is a partition with one part of size v. In order to prove the lemma we specify the partitions λ and ν as on the left hand side of (4.5.1). The partitions associated with the skew Schur function are

$$\lambda = (w^\alpha, u, 0^{\beta-1}) \tag{4.5.3}$$

and

$$\nu = (v, 0^{\alpha+\beta-1}). \tag{4.5.4}$$

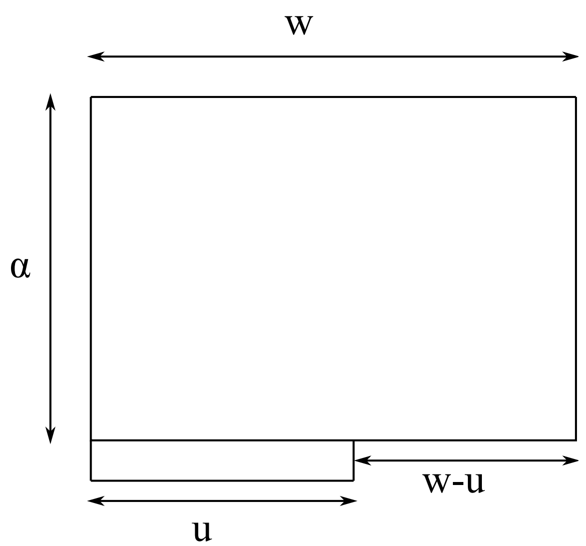

Figure 4.3: A diagram of the partition $\lambda = (w^\alpha, u, 0^{\beta-1})$ occurring in the identity (4.5.2).

The aim is to find an explicit expression for all partitions μ in the sum on the right hand side of (4.5.2). Given a partition λ of the shape depicted Figure 4.3, we want to find all partitions μ for which λ/μ is a horizontal strip of size v. This can be viewed as removing a strip of size v from λ so that the remaining object is still a valid partition. This removal can only be done from the last two rows, as removing anything from above the last two rows will not correspond to the removal of a strip. As the bottom row is of size u, the options of removing a strip of size v depend on the size of u and v. For this we consider two cases depending on whether the size v of the strip to be removed exceeds the length u of the bottom row or not.

Case $u \leq v$

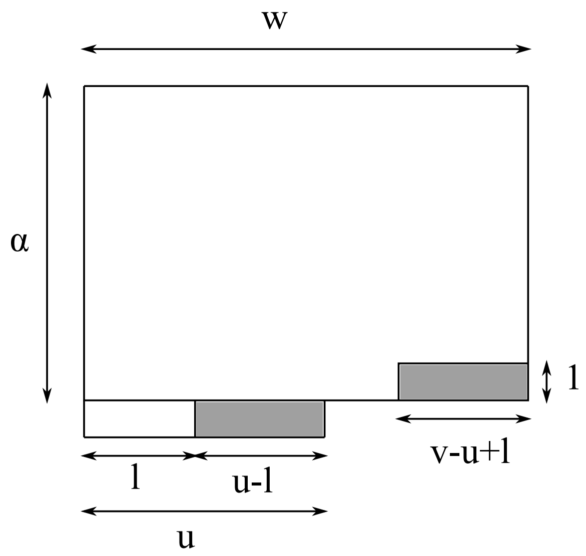

Figure 4.4: A diagram showing the structure of the partition $\mu = (w^{\alpha-1}, w-(v-u)-l, l)$ in the case $u \leq v$. The shaded part corresponds to a strip of size v.

Consider a skew partition given by $\lambda/\nu = (w^\alpha, u, 0^{\beta-1})/(v, 0^{\alpha+\beta-1})$ shown in Figure 4.2. If $u \leq v$ then the structure of the partitions μ appearing in the sum on the right hand side of (4.5.2) are indicated in Figure 4.4. The shaded portion shows the strip ν to be removed. We remove part of ν from the bottom row of length u and the remaining part from the row above, i.e. we shorten the bottom row by $u-l$

and the row above by $v-u+l$. We shall determine the allowed values of l below. Removing the strip ν from λ gives the following partition

$$\mu = (w^{\alpha-1}, w-(v-u)-l, l) \ . \tag{4.5.5}$$

Now we derive bounds for l. Firstly we see that $u-l \geq 0$. This implies

$$l \leq u. \tag{4.5.6}$$

Also we know that $v-u+l$ has to be less than or equal to $w-u$, else the grey parts will lie above each other and so we will no longer have a strip. From this condition we get

$$v-u+l \leq w-u \ ,$$

so

$$l \leq w-v \ . \tag{4.5.7}$$

From (4.5.6) and (4.5.7) it follows that

$$l \leq \min(u, w-v).$$

Also the condition $u < v$ implies that $w-u > w-v$. Therefore it follows that

$$0 \leq l \leq \min(u, v, w-u, w-v).$$

Therefore the sum can be written as

$$s_{(w^\alpha,u,0^{\beta-1})/(v,0^{\alpha+\beta-1})} = \sum_{l=0}^{\min(u,v,w-u,w-v)} s_{(w^{\alpha-1},w-(v-u)-l,l,0^{\beta-1})} \ . \tag{4.5.8}$$

Case $u > v$

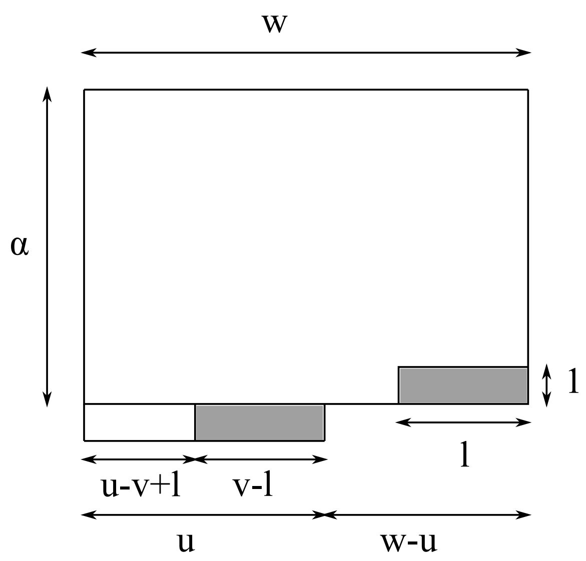

Figure 4.5: A diagram showing the structure of the partition $\mu = (w^{\alpha-1}, w-l, u-v+l)$ in the case $u > v$. The shaded part corresponds to a strip of size v.

We use the same idea as in the first case and remove strip ν from the partition λ. For $v < u$ the structure of the partitions μ appearing in the sum on the right hand side of (4.5.2) are indicated in Figure 4.5. Since $v < u$, we can remove ν completely from the lowest row and nothing from the row above, or we can remove part of it from the lowest row and the rest from the row above. We thus shorten the bottom row by $v - l$ and the row above by l. We shall determine the allowed values of l below. Removing the strip ν from λ therefore gives the partition

$$\mu = (w^{\alpha-1}, w-l, u-v+l) \ . \tag{4.5.9}$$

Consider the bounds for l. We see that $v - l \geq 0$, this implies

$$l \leq v \ . \tag{4.5.10}$$

Also we know that l has to be less than or equal to $w-u$, else the grey parts will lie above each other. From this condition we get

$$l \leq w-u \;. \tag{4.5.11}$$

From (4.5.10) and (4.5.11) we have

$$l \leq \min(v, w-u).$$

Also from the condition $v < u$ we have $w-u < w-v$. Then for l we have

$$0 \leq l \leq \min(u, v, w-u, w-v).$$

From the bounds for l and Schur partition μ we get

$$s_{(w^{\alpha},u,0^{\beta-1})/(v,0^{\alpha+\beta-1})} = \sum_{l=0}^{\min(u,v,w-u,w-v)} s_{(w^{\alpha-1},w-l,u-v+l,0^{\beta-1})} \tag{4.5.12}$$

We finally combine results (4.5.8) and (4.5.12). Denoting $r = \min(u, v, w-u, w-v)$, we find

$$s_{(w^{\alpha},u,0^{\beta-1})/(v,0^{\alpha+\beta-1})} = \sum_{l=0}^{r} s_{(w^{\alpha-1},w-(v-u)_{+}-l,(u-v)_{+}+l,0^{\beta-1})} \tag{4.5.13}$$

□

We now use this Lemma to state the desired equivalent result for Theorem 4.1 in terms of Schur functions. Note that while in Lemma 4.3 we did not need to specify the arguments of the functions, here it is important that the arguments are given by the kernel roots.

Corollary 4.4. The generating function $G^{w,\alpha,\beta}_{(u,v)}(t)$ of generalised weighted paths in terms of Schur functions is given by

$$G^{w,\alpha,\beta}_{(u,v)}(t) = (-1)^{1-\alpha}\frac{1}{tp_\alpha}\frac{\sum_{l=0}^{r} s_{(w^{\alpha-1},w-(v-u)_+-l,(u-v)_++l,0^{\beta-1})}(\bar{z})}{s_{((w+1)^\alpha,0^\beta)}(\bar{z})},$$

where $\bar{z}$ are the $\alpha+\beta$ roots of the kernel

$$K(t,z) = 1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B} q_b z^{-b}.$$

and $r = \min(u, v, w-u, w-v)$.

Proof. Lemma 4.3 proves the corollary. □

4.6 Examples

We now present several special cases involving small values of α and β. The first case we examine is $(\alpha,\beta) = (1,1)$, which corresponds to weighted Motzkin paths, and also includes Dyck paths as a special case, if the weight of the horizontal step is set to $p_0 = 0$. This has been studied previously [12] [6], but the Schur function approach used here is different and focusses more on the structure of the problem than just giving explicit generating functions. We then examine the cases $(\alpha,\beta) = (1,2)$ and $(\alpha,\beta) = (2,1)$, the solution of which involves roots of cubic equations. Here, the strength of our Schur function approach becomes apparent, as any explicit solution involves cumbersome algebraic expressions.

4.6.1 Motzkin paths

Theorem 4.1 shows that the geometric structure of the problem is encoded in the partition shapes, while the step weights are "hidden" in the kernel roots. For Motzkin paths the result is particularly simple and elegant, involving only partitions with two parts.

$$G^{w,1,1}_{(u,v)}(t) = \frac{1}{tp_1}\frac{s_{(w,u)/(v,0)}(z_1,z_2)}{s_{(w+1,0)}(z_1,z_2)} \, . \tag{4.6.1}$$

From a computational point of view, skew Schur functions are of course not that easy to evaluate, but with the help of Corollary 4.4 we are able to state the result in terms of Schur functions.

$$G^{w,1,1}_{(u,v)}(t) = \frac{1}{tp_1}\frac{\sum_{l=0}^{r} s_{(w-(v-u)_+-l,(u-v)_++l)}(z_1,z_2)}{s_{(w+1,0)}(z_1,z_2)}. \tag{4.6.2}$$

To expand the Schur functions we write them in terms of determinants. The Schur function in the denominator of Equation (4.6.2) is given by

$$\begin{aligned} s_{(w+1,0)}(z_1,z_2) =& \frac{1}{\Delta}\begin{vmatrix} z_1^{w+2} & z_2^{w+2} \\ z_1^0 & z_2^0 \end{vmatrix} & (4.6.3)\\ =& \frac{1}{\Delta}(z_1^{w+2} - z_2^{w+2}). & (4.6.4)\end{aligned}$$

where $\Delta = \Delta(z_1,z_2) = z_1 - z_2$ comes from a Vandermonde determinant evalution. Similarly expressing the Schur function in the numerator of Equation (4.6.2) as a determinant implies

$$\begin{aligned} s_{(w-(v-u)_+-l,(u-v)_++l)}(z_1,z_2) &= \frac{1}{\Delta}\begin{vmatrix} z_1^{w-(v-u)_+-l+1} & z_2^{w-(v-u)_+-l+1} \\ z_1^{(u-v)_++l} & z_2^{(u-v)_++l} \end{vmatrix} \\ &= \frac{1}{\Delta}(z_1^{w-(v-u)_+-l+1} z_2^{(u-v)_++l} - z_2^{w-(v-u)_+-l+1} z_1^{(u-v)_++l}). \quad (4.6.5)\end{aligned}$$

Now substituting the expansion of these Schur functions into (4.6.2), we finally obtain

$$G^{w,1,1}_{(u,v)}(t) = \frac{1}{tp_1}\frac{\sum_{l=0}^{\min(u,v,w-u,w-v)} (z_1^{w-(v-u)_+-l+1} z_2^{(u-v)_++l} - z_2^{w-(v-u)_+-l+1} z_1^{(u-v)_++l})}{z_1^{w+2} - z_2^{w+2}} \tag{4.6.6}$$

Here, $z_1 = z_1(t)$ and $z_2 = z_2(t)$ are the roots of the kernel $K(t, z) = 1 - tp_0 - tp_1 z - tq_1/z$, so that they can be explicitly given as solutions of the quadratic equation

$$z^2 - \frac{1/t - p_0}{p_1} z + \frac{q_1}{p_1} = 0 \; . \tag{4.6.7}$$

4.6.2 Case ($\alpha = 1$, $\beta = 2$)

Structurally, this case is rather similar to the preceding one, however the Schur functions now have as argument three kernel roots $z_1(t), z_2(t)$ and $z_3(t)$, which are the solution to the kernel equation given by

$$z^3 - \frac{1/t - p_0}{p_1} z^2 + \frac{q_1}{p_1} z + \frac{q_2}{p_1} = 0 \; , \tag{4.6.8}$$

so that a general explicit solution would involve roots of a cubic equation. Theorem 4.1 implies that

$$G^{w,1,2}_{(u,v)}(t) = \frac{1}{tp_1} \frac{s_{(w,u,0)/(v,0,0)}(z_1, z_2, z_3)}{s_{(w+1,0,0)}(z_1, z_2, z_3)} \; , \tag{4.6.9}$$

and the result given in Corollary 4.4 can be written as

$$G^{w,1,2}_{(u,v)}(t) = \frac{1}{tp_1} \frac{\sum_{l=0}^{r} s_{(w-(v-u)_+-l,(u-v)_++l,0)}(z_1, z_2, z_3)}{s_{(w+1,0,0)}(z_1, z_2, z_3)} \; . \tag{4.6.10}$$

We expand the Schur functions and write them in form of determinants. The Schur function in the denominator is given by

$$\begin{aligned} s_{(w+1,0,0)}(z_1, z_2, z_3) &= \frac{1}{\Delta} \begin{vmatrix} z_1^{w+3} & z_2^{w+3} & z_3^{w+3} \\ z_1^1 & z_2^1 & z_3^1 \\ z_1^0 & z_2^0 & z_3^0 \end{vmatrix} \\ &= \frac{1}{\Delta} \left(z_1^{w+3}(z_2 - z_3) - z_2^{w+3}(z_1 - z_3) + z_3^{w+3}(z_1 - z_2) \right) \; , \end{aligned} \tag{4.6.11}$$

where $\Delta = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$ is again a Vandermonde determinant (which will however cancel out in the final result). Similarly expressing the Schur function in the numerator as a determinant implies

$$
\begin{aligned}
&s_{(w-(v-u)_+-l,(u-v)_++l,0)}(z_1, z_2, z_3) = \\
&\qquad \frac{1}{\Delta}\Big(z_1^{w-(v-u)_+-l+2}\big(z_2^{(u-v)_++l+1} - z_3^{(u-v)_++l+1}\big) \\
&\qquad\qquad - z_2^{w-(v-u)_+-l+2}\big(z_1^{(u-v)_++l+1} - z_3^{(u-v)_++l+1}\big) \\
&\qquad\qquad\qquad + z_3^{w-(v-u)_+-l+2}\big(z_1^{(u-v)_++l+1} - z_2^{(u-v)_++l+1}\big)\Big) .
\end{aligned}
\tag{4.6.12}
$$

Now substituting the expansion of Schur functions in (4.6.10), we obtain

$$
\begin{aligned}
&G^{w,1,2}_{(u,v)}(t) = \\
&\frac{1}{tp_1}\frac{\displaystyle\sum_{l=0}^{\min(u,v,w-u,w-v)} \left(\begin{array}{l} z_1^{w-(v-u)_+-l+2}\big(z_2^{(u-v)_++l+1} - z_3^{(u-v)_++l+1}\big) \\ \quad - z_2^{w-(v-u)_+-l+2}\big(z_1^{(u-v)_++l+1} - z_3^{(u-v)_++l+1}\big) \\ \qquad + z_3^{w-(v-u)_+-l+2}\big(z_1^{(u-v)_++l+1} - z_2^{(u-v)_++l+1}\big) \end{array}\right)}{z_1^{w+3}(z_2 - z_3) - z_2^{w+3}(z_1 - z_3) + z_3^{w+3}(z_1 - z_2)} .
\end{aligned}
\tag{4.6.13}
$$

4.6.3 Case ($\alpha = 2$, $\beta = 1$)

The kernel equation now leads to

$$
z^3 + \frac{p_1}{p_2} z^2 - \frac{1/t - p_0}{p_2} z + \frac{q_1}{p_2} = 0 . \tag{4.6.14}
$$

We note that exchanging α and β is akin to switching up and down steps with adjusting the weights appropriately. More precisely, making all the parameters explicit we have

$$
K^{(2,1)}_{p_0,p_1,p_2,q_1}(t, z) = K^{(1,2)}_{p_0,q_1,q_2,p_1}(t, 1/z) , \tag{4.6.15}
$$

which in the case of unit weights implies that the kernel roots for $(\alpha,\beta)=(2,1)$ and $(\alpha,\beta)=(1,2)$ are simply inverses of each other. This symmetry is not as explicit when writing the generating functions in terms of Schur functions. Symmetry considerations would dictate that we need to replace u and v by $w-u$ and $w-v$, respectively, but this is not obvious from the result given in Theorem 4.1, which now reads

$$G^{w,2,1}_{(u,v)}(t) = -\frac{1}{tp_2}\frac{s_{(w,w,u)/(v,0,0)}(z_1,z_2,z_3)}{s_{(w+1,w+1,0)}(z_1,z_2,z_3)} \,. \tag{4.6.16}$$

From Corollary 4.4, this can be written as

$$G^{w,2,1}_{(u,v)}(t) = -\frac{1}{tp_2}\frac{\sum_{l=0}^{r} s_{(w,w-(v-u)_+-l,(u-v)_++l)}(z_1,z_2,z_3)}{s_{(w+1,w+1,0)}(z_1,z_2,z_3)} \,. \tag{4.6.17}$$

We expand the Schur functions and write them in form of determinants, and we obtain

$$G^{w,2,1}_{(u,v)}(t) = -\frac{1}{tp_2}\frac{\displaystyle\sum_{l=0}^{\min(u,v,w-u,w-v)}\left(\begin{array}{l} z_1^{w+2}\big(z_2^{w-(v-u)_+-l+1}z_3^{(u-v)_++l}-z_3^{w-(v-u)_+-l+1}z_2^{(u-v)_++l}\big)\\ \quad -z_2^{w+2}\big(z_1^{w-(v-u)_+-l+1}z_3^{(u-v)_++l}-z_3^{w-(v-u)_+-l+1}z_1^{(u-v)_++l}\big)\\ \quad +z_3^{w+2}\big(z_1^{w-(v-u)_+-l+1}z_2^{(u-v)_++l}-z_2^{w-(v-u)_+-l+1}z_1^{(u-v)_++l}\big)\end{array}\right)}{z_1^{w+3}\big(z_2^{w+2}-z_3^{w+2}\big)-z_2^{w+3}\big(z_1^{w+2}-z_3^{w+2}\big)+z_3^{w+3}\big(z_1^{w+2}-z_2^{w+2}\big)} \,. \tag{4.6.18}$$

When written in terms of kernel roots, we see some structural similarity between (4.6.18) and (4.6.13), in line with the symmetry observation made above.

Chapter 5

Adsorption of Generalised Weighted Paths at Boundaries

5.1 Introduction

Recall from Chapter 4 the model of generalised weighted paths. In this chapter we extend the generalised weighted paths by including adsorption of generalised paths at boundaries. Before considering paths with general step sets, we first present the model of Motzkin paths in Section 5.2 in order to set up the ideas for the general case, which is then treated in Section 5.3. We note that adsorption of Motzkin paths has been considered in [5]. Here we focus on adapting our methodology of Chapter 4 to the treatment of paths with boundary weights.

5.2 Motzkin paths

Motzkin paths are closely related to Dyck paths. The step set includes the steps from a Dyck path that is an up step $(1, 1)$ and a down step $(1, -1)$ and an addition of a horizontal step $(1, 0)$ which is what makes it different from Dyck paths. To be precise, we consider lifted Motzkin paths [27], where starting and end points are not restricted to be at height zero as is the case for Motzkin paths as normally defined. For the adsorption model consider the generating function of Motzkin paths as

$$M(\kappa, \lambda, t, z) = \sum_{\omega} \kappa^a \lambda^b z^v t^n \, , \tag{5.2.1}$$

where ω represents the set of all Motzkin paths in a slit of width $w > 0$ starting at a fixed height u with $0 \leq u \leq w$. We denote the total number of edges leaving or lying on the line $y = 0$ by a (this is equivalent to the number of vertices on the line with the exception of the final vertex), with b representing the total number of edges leaving or lying on the line $y = w$. v is the ending height of the path and n is the number of edges in a Motzkin path.

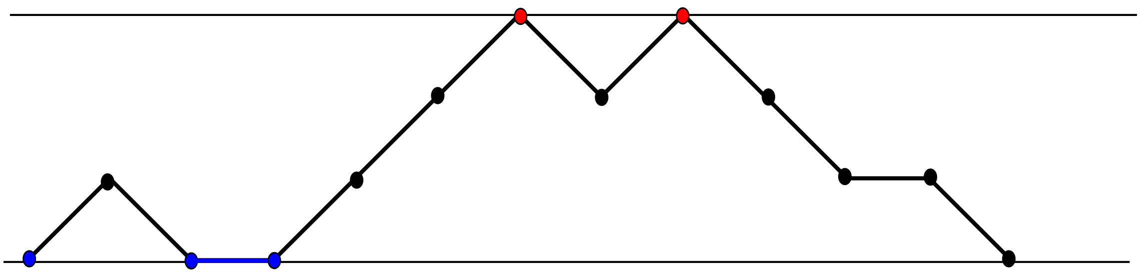

Figure 5.1: Motzkin path with edge and vertex visits with $a = 3$ and $b = 2$.

5.2.1 Functional equation

We construct a weighted path by appending steps from the step set of Motzkin paths to an $n - 1$ step path where $n > 0$. The up step is weighted by p, the down step by q and the horizontal step by r. This leads to the functional equation for the generating function $M(\kappa, \lambda, t, z)$,

$$\begin{aligned}M(\kappa, \lambda, t, z) = z^u + t(pz + r + \frac{q}{z})M(\kappa, \lambda, t, z) - \frac{tq}{z}M_0(\kappa, \lambda, t) - tpz^{w+1}M_w(\kappa, \lambda, t)\\ + tp(\kappa - 1)zM_0(\kappa, \lambda, t) + tq(\lambda - 1)z^{w-1}M_w(\kappa, \lambda, t)\\ + tr(\kappa - 1)M_0(\kappa, \lambda, t) + tr(\lambda - 1)z^w M_w(\kappa, \lambda, t)\,, \quad (5.2.2)\end{aligned}$$

where $M_0(\kappa, \lambda, t) = [z^0]M(\kappa, \lambda, t, z)$ and $M_w(\kappa, \lambda, t) = [z^w]M(\kappa, \lambda, t, z)$.

Here z^u represents the zero step walk starting and ending at height u. The term $t(pz + r + \frac{q}{z})M(\kappa, \lambda, t, z)$ corresponds to steps appended irrespective of whether the resulting walk steps leave the slit. The steps not allowed are removed by subtracting the terms which account for the steps crossing the boundaries $y = 0$, and $y = w$. For example $\frac{tq}{z}M_0(\kappa, \lambda, t)$ adjusts for steps going below the line $y = 0$, and $tpz^{w+1}M_w(\kappa, \lambda, t)$ adjusts for steps going beyond $y = w$. Next the term $tp(\kappa - 1)zM_0(\kappa, \lambda, t)$ adds a weight κ to an up step leaving $y = 0$. Similarly the terms $tr(\kappa - 1)M_0(\kappa, \lambda, t) + tr(\lambda - 1)z^w M_w(\kappa, \lambda, t)$ give a weight of κ and λ to horizontal steps at the boundaries, and lastly $tq(\lambda - 1)z^{w-1}M_w(\kappa, \lambda, t)$ gives a weight of λ to a down step leaving $y = w$.

To reduce notational overload, we now write $M(t, z) \equiv M(\kappa, \lambda, t, z)$ and analogously $M_v(t) = [z^v]M(\kappa, \lambda, t, z)$. Collecting the coefficients of $M(t, z)$, we rewrite the functional equation as

$$\begin{aligned}\left(1 - tpz - tr - t\frac{q}{z}\right) M(t, z) = z^u - \frac{tq}{z}M_0(t) - tpz^{w+1}M_w(t) + tp(\kappa - 1)zM_0(t)\\ + tq(\lambda - 1)z^{w-1}M_w(t) + tr(\kappa - 1)M_0(t) + tr(\lambda - 1)z^w M_w(t)\,. \quad (5.2.3)\end{aligned}$$

5.2.2 The kernel

The coefficient of $M(t, z)$ in functional equation (5.2.3) is the kernel given by

$$K(t, z) = 1 - tpz - tr - t\frac{q}{z}\,. \quad (5.2.4)$$

We rewrite the kernel as

$$K(t,z) = -\frac{tp}{z}\left(z^2 - \frac{1}{tp}z + \frac{r}{p}z + \frac{q}{p}\right). \tag{5.2.5}$$

As in Lemma 4.2. the kernel expressed in the terms of elementary symmetric functions is given by

$$K(t,z) = -\frac{tp}{z}\left(z^2 e_0 - z e_1 + e_2\right) = -tp\left(\sum_{i=0}^{2} z^{1-i}(-1)^i e_i\right). \tag{5.2.6}$$

We aim to relate this to our work in Chapter 4 and use the same methodology to solve for Motzkin paths under adsorption. For this we express our functional equation (5.2.3) in terms of elementary symmetric functions and then solve it.

5.2.3 Solution of the functional equation

The functional equation with the kernel in terms of elementary symmetric functions can be written as

$$\begin{aligned} -tp\left(\sum_{i=0}^{2} z^{1-i}(-1)^i e_i\right) M(t,z) &= z^u - \frac{tq}{z}M_0(t) - tpz^{w+1}M_w(t) + tp(\kappa-1)zM_0(t) \\ &+ tq(\lambda-1)z^{w-1}M_w(t) + tr(\kappa-1)M_0(t) + tr(\lambda-1)z^w M_w(t)\,. \end{aligned} \tag{5.2.7}$$

From this functional equation we want to derive equations for the coefficients $M_v(t)$. Writing $M(t,z) = \sum_{v=0}^{w} M_v(t) z^v$ and dividing throughout by $-tp$, we get

$$\left(\sum_{i=0}^{2} z^{1-i}(-1)^i e_i\right) \sum_{v=0}^{w} M_v(t) z^v = \frac{-1}{tp} z^u - \left(\frac{-1}{tp}\right) \frac{tq}{z} M_0(t) - \left(\frac{-1}{tp}\right) tpz^{w+1} M_w(t)$$
$$+ \left(\frac{-1}{tp}\right) tp(\kappa-1) z M_0(t) + \left(\frac{-1}{tp}\right) tq(\lambda-1) z^{w-1} M_w(t)$$
$$+ \left(\frac{-1}{tp}\right) tr(\kappa-1) M_0(t) + \left(\frac{-1}{tp}\right) tr(\lambda-1) z^w M_w(t)\,. \quad (5.2.8)$$

Using the result from Lemma 4.2, we replace the terms on the right hand side with elementary symmetric functions. We also rewrite the left hand side by changing the order of summation and extending the limits of the summation over v to infinity, as all the added terms are identically equal to zero. This transforms the functional equation as follows:

$$\sum_{v=-\infty}^{\infty} \left(\sum_{i=0}^{2} (-1)^i e_i M_{v+i-1}(t)\right) z^v = \frac{-1}{tp} z^u + \frac{e_2}{z} M_0(t) + e_0 z^{w+1} M_w(t) - e_0(\kappa-1) z M_0(t)$$
$$- e_2(\lambda-1) z^{w-1} M_w(t) - \frac{r}{p}(\kappa-1) M_0(t) - \frac{r}{p}(\lambda-1) z^w M_w(t)\,. \quad (5.2.9)$$

Terms on the right hand side with powers of z less than 0 and greater than w are cancelled out with the corresponding terms on the left hand side, and we are left with

$$\sum_{v=0}^{w} \left(\sum_{i=0}^{2} (-1)^i e_i M_{v+i-1}(t)\right) z^v = \frac{-1}{tp} z^u - e_0(\kappa-1) z M_0(t)$$
$$- e_2(\lambda-1) z^{w-1} M_w(t) - \frac{r}{p}(\kappa-1) M_0(t) - \frac{r}{p}(\lambda-1) z^w M_w(t)\,. \quad (5.2.10)$$

Comparing coefficients of z^v for $0 \le v \le w$, Equation (5.2.10) is equivalent to a system of $w+1$ equations for $M_{(u,v)}(t) = M_v(t)$, where we now have made the starting height u explicit in our notation.

$$\begin{bmatrix} \left(-e_1+\frac{r}{p}(\kappa-1)\right) & e_2 & 0 & \cdots & 0 \\ e_0\kappa & -e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & -e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2\lambda \\ 0 & 0 & 0 & \cdots & \left(-e_1+\frac{r}{p}(\lambda-1)\right) \end{bmatrix} \begin{bmatrix} M_{(u,0)}(t) \\ M_{(u,1)}(t) \\ M_{(u,2)}(t) \\ \vdots \\ M_{(u,w)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ -\frac{1}{tp} \\ \vdots \\ 0 \end{bmatrix}. \tag{5.2.11}$$

Here $-\frac{1}{tp}$ in the right hand column vector is an entry at the u^{th} position, with every other entry being zero. For readability we introduce κ' and λ' by letting $-e_1\kappa' = -e_1 + \frac{r}{p}(\kappa - 1)$ and $-e_1\lambda' = -e_1 + \frac{r}{p}(\lambda - 1)$. Equation (5.2.11) thus gives the following simple system of equations

$$\begin{bmatrix} -e_1\kappa' & e_2 & 0 & \cdots & 0 \\ e_0\kappa & -e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & -e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2\lambda \\ 0 & 0 & 0 & \cdots & -e_1\lambda' \end{bmatrix} \begin{bmatrix} M_{(u,0)}(t) \\ M_{(u,1)}(t) \\ M_{(u,2)}(t) \\ \vdots \\ M_{(u,w)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ -\frac{1}{tp} \\ \vdots \\ 0 \end{bmatrix}, \tag{5.2.12}$$

which differs from (4.4.11) by the inclusion of boundary weights κ, κ', λ, and λ' in the first and last column.

We now again use Cramer's rule to evaluate the unknowns $M_{(u,v)}(t)$. To be able to express the resulting determinants using Jacobi-Trudi formulas, it is convenient to eliminate the negative signs in front of the elementary symmetric functions. A transformation is applied by multiplying with the matrix S as we did in Chapter 4, where the transformed matrix equation was $A(Sx) = (-1)^{u+1-\alpha}b$. For Motzkin paths α is simply equal to 1 and so we find

$$\begin{bmatrix} e_1\kappa' & e_2 & 0 & \cdots & 0 \\ e_0\kappa & e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2\lambda \\ 0 & 0 & 0 & \cdots & e_1\lambda' \end{bmatrix} \begin{bmatrix} M_{(u,0)}(t) \\ -M_{(u,1)}(t) \\ M_{(u,2)}(t) \\ \vdots \\ (-1)^w M_{(u,w)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ (-1)^u \frac{1}{tp} \\ \vdots \\ 0 \end{bmatrix}. \tag{5.2.13}$$

The aim is to solve this matrix equation using Cramer's rule. The unknowns are given by

$$(-1)^v M_{(u,v)}(t) = \frac{|A_{(u,v)}|}{|A|}, \tag{5.2.14}$$

where A is given by

$$A = \begin{bmatrix} e_1\kappa' & e_2 & 0 & \cdots & 0 \\ e_0\kappa & e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2\lambda \\ 0 & 0 & 0 & \cdots & e_1\lambda' \end{bmatrix}, \tag{5.2.15}$$

and $A_{(u,v)}$ is the minor obtained by replacing the v^{th} column in A by the column vector on the right hand side of equation (5.2.13). To evaluate the determinant of A, we use generalised cofactor expansion. The expansion is done by the first and last column of the matrix A. We only consider the case $w > 2$ here, as smaller values of w can be easily calculated directly without recourse to the formalism used below. Generally the expansion gives a sum over $\binom{w+1}{2}$ different terms, but due to the presence of zeros in the first and last column most of these terms are zero and we are left with a sum over four terms only. We find

$$|A| = \begin{vmatrix} e_1\kappa' & 0 \\ 0 & e_1\lambda' \end{vmatrix} \begin{vmatrix} e_1 & e_2 & 0 & \cdots & 0 \\ e_0 & e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2 \\ 0 & 0 & 0 & \cdots & e_1 \end{vmatrix} - \begin{vmatrix} e_1\kappa' & 0 \\ 0 & e_2\lambda \end{vmatrix} \begin{vmatrix} e_1 & e_2 & 0 & \cdots & 0 \\ e_0 & e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2 \\ 0 & 0 & 0 & \cdots & e_0 \end{vmatrix}$$

$$- \begin{vmatrix} e_0\kappa & 0 \\ 0 & e_1\lambda' \end{vmatrix} \begin{vmatrix} e_2 & 0 & 0 & \cdots & 0 \\ e_0 & e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2 \\ 0 & 0 & 0 & \cdots & e_1 \end{vmatrix} + \begin{vmatrix} e_0\kappa & 0 \\ 0 & e_2\lambda \end{vmatrix} \begin{vmatrix} e_2 & 0 & 0 & \cdots & 0 \\ e_0 & e_1 & e_2 & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_2 \\ 0 & 0 & 0 & \cdots & e_0 \end{vmatrix}, \tag{5.2.16}$$

where the sign is determined by evaluating the sign of the corresponding permutations used in (2.7.3). Using the notation in that equation, $H = \{1, w+1\}$, and L is one of $\{1, w\}$, $\{1, w+1\}$, $\{2, w\}$ and $\{2, w+1\}$. Using the Jacobi-Trudi formula for each individual determinant we can express the determinant of A in terms of Schur functions as

$$|A| = {e_1}^2\kappa'\lambda's_{(w-1,0)} - e_1e_2\kappa'\lambda s_{(w-2,0)} - e_0e_1\kappa\lambda's_{(w-1,1)} + e_0e_2\kappa\lambda s_{(w-2,1)}\,. \tag{5.2.17}$$

This concludes our evaluation of the determinant of A and we next turn to the evaluation of the determinant of $A_{(u,v)}$. To avoid degenerate cases, we shall restrict ourselves to the case $1 \le v \le w-1$ and $2 \le u \le w-2$, so in particular, we need $w > 3$ here. As per Cramer's rule we substitute the column vector on right hand side of equation (5.2.13) into A, leading to

$$A_{(u,v)} = \begin{bmatrix} e_1\kappa' & e_2 & 0 & \cdots & 0 & \cdots & 0 \\ e_0\kappa & e_1 & e_2 & \cdots & 0 & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^u \frac{1}{tp} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & e_2\lambda \\ 0 & 0 & 0 & \cdots & 0 & \cdots & e_1\lambda' \end{bmatrix}. \tag{5.2.18}$$

To compute the determinant of $A_{(u,v)}$ we expand by the v^{th} column and get

$$A_{(u,v)} = (-1)^{u+v} \frac{(-1)^u}{tp} \begin{vmatrix} e_1\kappa' & e_2 & 0 & \cdots & \cdots & 0 \\ e_0\kappa & e_1 & e_2 & \cdots & \cdots & 0 \\ 0 & e_0 & e_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & e_2\lambda \\ 0 & 0 & 0 & \cdots & \cdots & e_1\lambda' \end{vmatrix}. \tag{5.2.19}$$

Using generalised Laplace expansion along the first and the last column, we obtain

$$\begin{aligned} A_{(u,v)} = (-1)^{u+v} \frac{(-1)^u}{tp} \\ \left(\begin{vmatrix} e_1\kappa' & 0 \\ 0 & e_1\lambda' \end{vmatrix} D_1 - \begin{vmatrix} e_1\kappa' & 0 \\ 0 & e_2\lambda \end{vmatrix} D_2 - \begin{vmatrix} e_0\kappa & 0 \\ 0 & e_1\lambda' \end{vmatrix} D_3 + \begin{vmatrix} e_0\kappa & 0 \\ 0 & e_2\lambda \end{vmatrix} D_4 \right) \end{aligned} \tag{5.2.20}$$

where D_1, D_2, D_3 and D_4 are the complementary minors. Using the second Jacobi-

Trudi identity, we evaluate the determinants of each of the minors and get

$$|A_{(u,v)}| = (-1)^{u+v} \frac{(-1)^u}{tp} \left({e_1}^2 \kappa' \lambda' s_{(w-2,u-1)/(v-1)} - e_1 e_2 \kappa' \lambda s_{(w-3,u-1)/(v-1)} \right. \\ \left. - e_0 e_1 \kappa \lambda' s_{(w-2,u-1,1)/(v-1)} + e_0 e_2 \kappa \lambda s_{(w-3,u-1,1)/(v-1)} \right) . \quad (5.2.21)$$

Note that this expression ceases to make sense for u or v close to the boundary, as expected. Also, we dropped the inclusion of zero size parts in the partition that we used when writing Schur functions as it makes less sense here. We get the unknown generating function by substituting the determinants in (5.2.14). This cancels the signs and gives

$$M_{(u,v)}(t) = \frac{1}{tp} \frac{\begin{pmatrix} {e_1}^2 \kappa' \lambda' S_{(w-2,u-1)/(v-1)} - e_1 e_2 \kappa' \lambda S_{(w-3,u-1)/(v-1)} \\ - e_0 e_1 \kappa \lambda' S_{(w-2,u-1,1)/(v-1)} + e_0 e_2 \kappa \lambda S_{(w-3,u-1,1)/(v-1)} \end{pmatrix}}{{e_1}^2 \kappa' \lambda' S_{(w-1)} - e_1 e_2 \kappa' \lambda S_{(w-2)} - e_0 e_1 \kappa \lambda' S_{(w-1,1)} + e_0 e_2 \kappa \lambda S_{(w-2,1)}} . \quad (5.2.22)$$

This completes the computation of the generating function of Motzkin paths under adsorption.

5.3 Generalised weighted paths under adsorption

Here we will use the machinery developed for Motzkin paths for the adsorption of generalised weighted paths. We will again arrive at a matrix equation that can be solved in terms of skew Schur functions.

5.3.1 Generating function and functional equation

Consider the generalised weighted paths introduced in Chapter 4. We add boundary weights κ and λ analogously to what we did for Motzkin paths, so that in extension of the notation in (4.1.1), the generating function is now given by

$$G(t,z) = \sum_{v=0}^{w} G^{w,\alpha,\beta}_{(u,v)}(\kappa,\lambda;t) z^v \,, \tag{5.3.1}$$

where the generating function representing the paths ending at some height v (and starting at fixed height u) is given by

$$G_v(t) = G^{w,\alpha,\beta}_{(u,v)}(\kappa,\lambda;t)\,. \tag{5.3.2}$$

The functional equation for the generating function $G(t,z)$ is now given by

$$\begin{aligned} G(t,z) = z^u + t\left(\sum_{a\in A} p_a z^a + \sum_{b\in B} \frac{q_b}{z^b}\right) G(t,z) \\ - t\sum_{j=1}^{\infty} z^{w+j} \sum_{a\geq j} p_a G_{w-a+j}(t) - t\sum_{j=1}^{\infty} z^{-j} \sum_{b\geq j} q_b G_{b-j}(t) \\ + tp_0(\kappa-1)G_0(t) + tp_0(\lambda-1)z^w G_w(t) + \sum_{a=1}^{\alpha} tp_a(\kappa-1)z^a G_0(t) + \sum_{b=1}^{\beta} tq_b(\lambda-1)z^{w-b} G_w(t)\,, \end{aligned} \tag{5.3.3}$$

where the last four terms account for boundary weights. Note that there is a correction for the horizontal step with weight p_0 at both the top and bottom border. The kernel of the functional equation

$$K(t,z) = 1 - t\sum_{a\in A} p_a z^a - t\sum_{b\in B} \frac{q_b}{z^b} \tag{5.3.4}$$

is identical to the one considered in Chapter 4.

5.3.2 Solution of the functional equation

The same manipulations as in the case for Motzkin paths now lead to a slightly more involved Matrix equation. From the functional equation we find

$$\sum_{v=0}^{w}\left(\sum_{i=0}^{\alpha+\beta}(-1)^i e_i G_{v-\alpha+i}(t)\right) z^v = -\frac{z^u}{tp_\alpha} - \frac{p_0}{p_\alpha}(\kappa-1)G_0(t) - \frac{p_0}{p_\alpha}(\lambda-1)z^w G_w(t)$$
$$-\frac{1}{p_\alpha}\sum_{a=1}^{\alpha} p_a(\kappa-1)z^a G_0(t) - \frac{1}{p_\alpha}\sum_{b=1}^{\beta} q_b(\lambda-1)z^{w-b}G_w(t)\,. \quad (5.3.5)$$

As the next steps in the calculation are indentical to what has been done above, we shall only give the matrix equation after the change of signs. Here we use κ' and λ' defined via $(-1)^\alpha e_\alpha \kappa' = (-1)^\alpha e_\alpha + \frac{p_0}{p_\alpha}(\kappa-1)$ and $(-1)^\alpha e_\alpha \lambda' = (-1)^\alpha e_\alpha + \frac{p_0}{p_\alpha}(\lambda-1)$.

$$(-1)^\alpha \begin{bmatrix} e_\alpha\kappa' & e_{\alpha+1} & \cdots & e_{\alpha+\beta} & \cdots & 0 \\ e_{\alpha-1}\kappa & e_\alpha & \cdots & e_{\alpha+\beta-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_0\kappa & e_1 & \cdots & e_\beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e_\alpha\lambda' \end{bmatrix} \begin{bmatrix} G_{(u,0)}(t) \\ -G_{(u,1)}(t) \\ G_{(u,2)}(t) \\ \vdots \\ (-1)^w G_{(u,w)}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ (-1)^{u+1}\frac{1}{tp_\alpha} \\ \vdots \\ 0 \end{bmatrix}. \quad (5.3.6)$$

Using Cramer's rule we proceed with computing $G_{(u,v)}(t)$ from

$$(-1)^v G_{(u,v)}(t) = \frac{|A_{(u,v)}|}{|A|}\,. \quad (5.3.7)$$

We use a generalised cofactor expansion with respect to the first and last column and write the resulting cofactors in terms of Schur functions using Jacobi-Trudi formulas. We only give the result away from the boundaries, restricting the range of u and v to satisfy $0 < v < w$ and $\alpha < u < w - \beta$. With the notation

$$\kappa_i = \begin{cases} \kappa' & i = 0 \\ \kappa & i > 0 \end{cases} \quad \text{and} \quad \lambda_j = \begin{cases} \lambda' & j = 0 \\ \lambda & j > 0 \end{cases}, \tag{5.3.8}$$

we find that

$$|A| = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} (-1)^{i+j} \kappa_i \lambda_j e_{\alpha-i} e_{\alpha+j} S_{(w-1^{\alpha-1}, w-1-j, i, 0^{\beta-1})} \tag{5.3.9}$$

and

$$|A_{(u,v)}| = \frac{(-1)^{\alpha+v+1}}{tp_\alpha} \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} (-1)^{i+j} \kappa_i \lambda_j e_{\alpha-i} e_{\alpha+j} S_{(w-2^{\alpha-1}, w-2-j, u-1)/(v-1)} . \tag{5.3.10}$$

Our final result is therefore given as

$$G_{(u,v)}(t) = \frac{(-1)^{\alpha+1}}{tp_\alpha} \frac{\sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} (-1)^{i+j} \kappa_i \lambda_j e_{\alpha-i} e_{\alpha+j} S_{(w-2^{\alpha-1}, w-2-j, u-1)/(v-1)}}{\sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} (-1)^{i+j} \kappa_i \lambda_j e_{\alpha-i} e_{\alpha+j} S_{(w-1^{\alpha-1}, w-1-j, i, 0^{\beta-1})}} . \tag{5.3.11}$$

We note that there is a slight inconsistency in the Schur and skew Schur function notations we have used in this Thesis. For Schur functions, we preferred to indicate the dimension explicitly by giving a partition with $\alpha + \beta$ parts, supplementing with zeros if necessary. We continued to do this for skew Schur functions in Lemma 4.3. However, we saw already in the Motzkin path case, where $\alpha + \beta = 2$, the number of parts in the skew Schur function increased to 3, so that our preferred notation became less useful. We therefore dropped the trailing zeros when indicating the parts of the partition in the skew Schur function notation.

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